COMPARATIVE STATICS UNDER UNCERTAINTY: SINGLE CROSSING PROPERTIES AND LOG-SUPERMODULARITY

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ABSTRACT: This paper develops necessary and sufficient conditions for monotone comparative statics predictions in several classes of stochastic optimization problems. The results are formulated so as to highlight the tradeoffs between assumptions about payoff functions and assumptions about probability distributions; they characterize "minimal sufficient conditions" on a pair of functions (for example, a utility function and a probability distribution) so that the expected utility satisfies necessary and sufficient conditions for comparative statics predictions. The paper considers two main classes of assumptions on primitives: single crossing properties and log-supermodularity. Single crossing properties arise naturally in portfolio investment problems and auction games. Log-supermodularity is closely related to several commonly studied economic properties, including decreasing absolute risk aversion, affiliation of random variables, and the monotone likelihood ratio property. The results are used to extend the existing literature on investment problems and games of incomplete information, including auction games and pricing games.

KEYWORDS: Monotone comparative statics, single crossing, affiliation, monotone likelihood ratio, log-supermodularity, investment under uncertainty, risk aversion.

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1. Introduction

Since Samuelson, economists have studied and applied systematic tools for deriving comparative statics predictions. Recently, the theory of comparative statics has received renewed attention (Topkis (1978), Milgrom and Roberts (1990a, 1990b, 1994), Milgrom and Shannon (1994)), and two main themes have emerged. First, the new literature stresses the utility of having general and widely applicable theorems about comparative statics. Second, this literature emphasizes the role of robustness of conclusions to changes in the specification of models, searching for the weakest possible conditions which guarantee that a comparative statics conclusion holds across a family of models. The literature shows that many of the robust comparative statics results that arise in economic theory rely on three main properties: supermodularity,[1] log-supermodularity, and single crossing properties.

In this paper, we study comparative statics in stochastic optimization problems, focusing on characterizations of single crossing properties and log-supermodularity (Athey (1995) studies supermodularity in stochastic problems). Two main classes of applications motivate the results in this paper. The first concerns investment under uncertainty, where an agent makes a risky investment, and her choice varies with her risk preferences or the distribution over the uncertain returns. Second, the paper considers games of incomplete information, such as pricing games and first price auctions, where the goal is to find conditions under which a player's action is nondecreasing in her (privately observed) type. One reason such a finding is useful is that Athey (1997) shows that whenever such a comparative statics result holds, a pure strategy Nash equilibrium (PSNE) will exist.

The theorems in this paper are designed to highlight the issues involved in selecting between different assumptions about a given problem, focusing on the tradeoffs between assumptions about payoff functions and probability distributions. The general class of problems under consideration can be written $U(\mathbf{x},q) = \quad u(\mathbf{x},\mathbf{s})f(\mathbf{s},q)dm(\mathbf{s})$, where each bold variable is a vector of real numbers, $\mathbf{x}$ represents a choice vector, and q represents an exogenous parameter. For several classes of restrictions on the primitives (u and f) which arise in the literature, this paper derives necessary and sufficient conditions for a comparative statics conclusion, that is, the conclusion that the choices $\mathbf{x}$ are nondecreasing with q.

The analysis builds on the work of Milgrom and Shannon (1994), who show that a necessary condition for the optimal choice of $\mathbf{x}$, $\mathbf{x}^*(q)$, to be nondecreasing in θ is that $U(\mathbf{x},q)$ satisfies the single crossing property in $(\mathbf{x};q)$. In this paper, we show if we desire that comparative statics hold not just for a particular utility function, but for all utility functions in a given class, it will often be

[1] A positive function on a product set is supermodular if increasing any one variable increases the returns to increasing each of the other variables; for differentiable functions this corresponds to non-negative cross-partial derivatives between each pair of variables. A function is log-supermodular if the log of that function is supermodular. See Section 2 for formal definitions.

necessary that $U(\mathbf{x},q)$ satisfies a stronger condition than single crossing. Once the desired property of $U(\mathbf{x},q)$ is identified, it remains to find the "best" pair of sufficient conditions on u and f to generate the comparative statics predictions. Most theorems in the paper are stated in terms of what we call "minimal pairs of sufficient conditions": they provide sufficient conditions on u and f for the comparative statics conclusion, and further, neither of the conditions can be weakened without strengthening the condition on the other primitive.

We begin by studying log-supermodular (henceforth abbreviated log-spm) problems. Log-spm primitives arise naturally in many contexts: for example, an agent's marginal utility $u'(w+s)$ is log-spm in (w,s) if the utility function satisfies decreasing absolute risk aversion, and $D(p;\varepsilon)$ is log-spm if demand becomes less elastic as ε increases. A density is log-spm if it has the property *affiliation* from auction theory or if it satisfies the *monotone likelihood ratio property*.

Log-supermodularity is a very convenient property for working with integrals, because (i) products of log-spm functions are log-spm, and (ii) integrating with respect to a log-spm density preserves both log-spm and single crossing properties. However, a critical feature for our analysis of log-supermodularity and the single crossing property in stochastic problems is that (unlike supermodularity), these properties are *not* preserved by arbitrary positive combinations.

The first main comparative statics theorem of the paper is established in two steps. We begin by showing that log-supermodularity of $U(\mathbf{x},q)$ is necessary to ensure that the optimal choices of $\mathbf{x}$ increase in response to exogenous changes in q *for all* payoffs u which are log-spm. We then establish necessary and sufficient conditions on the density f to guarantee this conclusion: f must be log-spm as well. The results can be applied to establish relationships between several commonly used orders over distributions in investment theory; they can also be used to show that decreasing absolute risk aversion is preserved when independent or affiliated background risks are introduced. We further consider a more novel application, a pricing game between multiple (possibly asymmetric) firms with private information about their marginal costs. The analysis characterizes conditions under which each firm's price increases in its marginal cost. Finally, we show that the results can be used to extend and unify some existing results (Milgrom and Weber, 1982; Whitt (1982)) about monotonicity of conditional expectations; these results are applied to derive comparative statics predictions when payoffs are supermodular.

Our second set of comparative statics theorems concern problems with a single random variable, where one of the primitive functions satisfies a single crossing property and the other is log-spm. We study necessary and sufficient conditions for the preservation of single crossing properties, and thus comparative statics results, under uncertainty. The results are extended in order to exploit the additional structure imposed by portfolio and investment problems. Further, in a first price auction with heterogeneous players and common values, the results imply conditions under which a player's optimal bid is nondecreasing in her signal. Finally, in problems of the form

$v(x,y,s)dF(s,q)$, we characterize single crossing of the x-y indifference curves; the results are applied to signaling games and consumption-savings problems.

The results in this paper build directly on a set of results from the statistics literature, which concerns "totally positive kernels," where for the case of strictly positive bivariate function, total positivity of order 2 (TP-2) is equivalent to log-spm. Lehmann (1955) proved that bivariate log-supermodularity is preserved by integration, while Karlin and Rubin (1956) studied the preservation of single crossing properties under integration with respect to log-spm densities. A number of papers in the statistics literature have exploited this relationship, and Karlin's (1968) monograph presents the general theory of the preservation of an arbitrary number of sign changes under integration. Following this, Ahlswede and Daykin (1979) extended the theory to multivariate functions, showing that multivariate total positivity of order 2 (MTP-2) is preserved by integration. Karlin and Rinott (1980) present the theory of MTP-2 functions together with a variety of useful applications, though comparative statics results are not among them. Thus, it is somewhat surprising that results developed primarily for other applications have such power in the study of comparative statics. Only a few papers in economics have exploited the results of this literature. Milgrom and Weber (1982) independently discovered many of the features of log-spm densities in their study of affiliated random variables in auctions. Jewitt (1987) exploits the work on the preservation of single crossing properties and bivariate log-spm in his analysis of orderings over risk aversion and associate comparative statics; he makes use of the fact that orderings over risk aversion can be recast as statements about log-spm of a marginal utility.[2]

This paper extends the existing literature in several ways. The results from the statistics literature identify a set of basic mathematical properties of stochastic problems. This paper customizes the results for application to the analysis of comparative statics in economic problems, especially in relation to the recent lattice-theoretic approach, and develops the remaining mathematical results required to provide a set of theorems about "minimal pairs" of sufficient conditions for comparative statics results. Each theorem is illustrated with economic applications, several of which themselves represent new results. Further, we identify the limits of the necessity results: we formally identify the smallest class of functions that must be admissible to derive necessary conditions. The applications are chosen to highlight the extent to which the additional structure available in economic problems allows the stated conditions to be weakened, and they motivate several extensions to the basic theorems that exploit additional structure.

In the economics literature, many authors have studied comparative statics in stochastic optimization problems in the context of classic investment under uncertainty problems,[3] asking

[2] See also Athey et al (1994), who apply the sufficiency conditions to an organizational design problem.

[3] Some notable contributions include Diamond and Stiglitz (1974), Eeckhoudt and Gollier (1995), Gollier (1995), Jewitt (1986, 1987, 1988b, 1989), Kimball (1990, 1993), Landsberger and Meilijson (1990), Meyer and Ormiston

questions about how an agent’s investment decisions change with the nature of the risk or uncertainty in the economic environment, or with the characteristics of the utility function. This paper shows how many of the results of the existing literature can be incorporated in a unified framework with fewer simplifying assumptions, and it suggests some new extensions.

This work is also related to Vives (1990) and Athey (1995), who study supermodularity in stochastic problems.[4] Vives (1990) shows that supermodularity is preserved by integration (which follows since arbitrary sums of supermodular functions are supermodular). Athey (1995) also studies properties which are preserved by convex combinations (including monotonicity and concavity), showing that all theorems about monotonicity of $\int u(\mathbf{s}) f(\mathbf{s};\theta) ds$ in correspond to analogous results about other properties “*P*” of $\int u(\mathbf{s}) f(\mathbf{s};\theta) ds$ in θ, in the case where “*P*” is a property which is preserved by convex combinations, and further u is in a closed convex set of payoff functions. However, the necessary and sufficient condition for comparative statics conclusions emphasized in this paper is a single crossing property, which is *not* preserved by convex combinations; thus, different approaches are required in this paper.[5] Independently, Gollier and Kimball (1995a, b) have also exploited convex analysis, developing several unifying theorems for the study of the comparative statics associated with risk.

The paper proceeds as follows. Section 2 introduces definitions. Section 3 explores log-spm problems, while Section 4 focuses on single crossing properties in problems with a single random variable. Section 5 concludes and presents Table 1, which summarizes the six main theorems about comparative statics proved in this paper. Proofs not stated in the text are in the Appendix.

2. Definitions

The following two sections introduce the main classes of properties to be studied in this paper, and links them to comparative statics theorems. It turns out that some of the same properties, single crossing properties, supermodularity and log-supermodularity, arise both as properties of primitives, and as the properties of stochastic objective functions which are equivalent to comparative statics predictions.

2.1. Single Crossing Properties

We will be concerned with a variety of different single crossing properties, detailed below.

Definition 1 *Let $g:\Re\rightarrow\Re$. (i) g(t) satisfies weak single crossing about a point t_0, denoted* ***WSC1(t_o)***, *if $g(t)\leq 0$ for all $t<t_0$, and $g(t)\geq 0$ for all $t>t_0$, while g(t) satisfies weak single crossing*

(1983, 1985, 1989), Ormiston (1992), and Ormiston and Schlee (1992, 1993); see Scarsini (1994) for a survey of the main results involving risk and risk aversion.

[4] See also Hadar and Russell (1978) and Ormiston (1992), who show that the theory of stochastic dominance can be used to derive comparative statics results based on supermodularity.

[5] The single crossing property is only preserved under convex combinations if the “crossing point” is fixed; see Section 5 and Athey (1998) for more discussion of this point.

*(**WSC1**) if there exists a t_0 such that g satisfies WSC1(t_0).*
*(ii) g(t) satisfies single crossing (**SC1**) in t if there exist $t_0' \leq t_0''$ such that $g(t)<0$ for all $t<t_0'$, $g(t)=0$ for all $t_0'<t<t_0''$, and $g(t)>0$ for all $t>t_0''$.*
*(iii) h(x,t) satisfies single crossing in two variables (**SC2**) in (x;t) if, for all $x_H>x_L$, $g(t)=h(x_H;t)-h(x_L;t)$ satisfies SC1. WSC2(t_0) and WSC2 are defined analogously.*
*(iv) h(x,y,t) satisfies single crossing in three variables (**SC3**) if h(x,b(x),t) satisfies SC2 for all $b:\Re\rightarrow\Re$.*

The definition of SC1 simply says that $g(t)$ crosses zero, at most once, from below (Figure 1). *Weak* SC1 allows the function g to return to 0 after it becomes positive. Single crossing in two variables, SC2, requires that the *incremental* returns to x cross zero at most once, from below, as a function of t; it is used to derive comparative statics results in non-differentiable problems or problems which are not quasi-concave. Milgrom and Shannon (1994) show that the set $\arg\max_{x\in B} h(x,t)$ is monotone nondecreasing in t for all B if and only if h satisfies SC2 in $(x;t)$;[6] this result will be applied repeatedly throughout the paper.[7] Finally, Milgrom and Shannon (1994) show that for suitably well-behaved functions, SC3 is equivalent to the Spence-Mirrlees single crossing property, that is, $h_1(x,y,t)/|h_2(x,y,t)|$ nondecreasing in t.

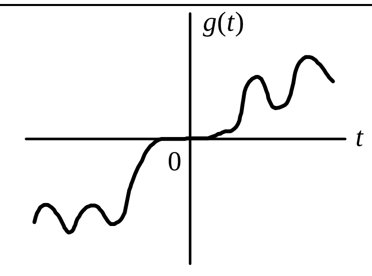

Figure 1: $g(t)$ satisfies SC1 in t.

2.2. Log-Supermodularity and Related Properties

For bivariate functions, supermodularity and log-supermodularity are stronger than SC2. Supermodularity requires that the incremental returns to increasing x, $g(t)=h(x_H;t)-h(x_L;t)$, must be nondecreasing (rather than SC1) in t; log-supermodularity of a positive function requires that the relative returns, $h(x_H;t)/h(x_L;t)$, are nondecreasing in t. Thus, as properties of objective functions, both are sufficient for comparative statics predictions. The multivariate version of supermodularity simply requires that the relationship just described holds for each pair of variables. Topkis (1978) proves that if h is twice differentiable, h is supermodular if and only if $\frac{\partial^2}{\partial x_i \partial x_j} h(\mathbf{x}) \geq 0$ for all $i \neq j$. If h is positive, then h is log-spm if and only if $\log(h(\mathbf{x}))$ is supermodular.

The definitions can be stated more generally using lattice theoretic notation. Given a set X and a partial order $\geq$, the operations "meet" ($\vee$) and "join" ($\wedge$) are defined as follows: $\mathbf{x}\vee\mathbf{y}=\inf\{\mathbf{z}|\mathbf{z}\geq\mathbf{x}, \mathbf{z}\geq\mathbf{y}\}$ and $\mathbf{x}\wedge\mathbf{y}=\sup\{\mathbf{z}|\mathbf{z}\leq\mathbf{x}, \mathbf{z}\leq\mathbf{y}\}$ (for $\Re^n$ with the usual order, these

[6] Notice that the comparative statics conclusion involves a quantification over constraint sets. This is because SC2 is a requirement about *every* pair of choices of x. Many of the comparative statics theorems of this paper eliminate the quantification over constraint sets. Instead, the theorems require the comparative statics result to hold across a class of models, where that class is sufficiently rich so that a global condition is required for a robust conclusion.

[7] Shannon (1995) establishes a related result for WSC2, namely, WSC2 is necessary and sufficient for the existence of a nondecreasing optimizer.

represent the component-wise maximum and component-wise minimum, respectively). A lattice is a set X together with a partial order, such that the set is closed under meet and join.

Definition 2 *Let* $(X,\geq)$ *be a lattice. A function* $h{:}X\rightarrow\Re$ *is* ***supermodular*** *if, for all* $\mathbf{x},\mathbf{y}\in X$, $h(\mathbf{x}\vee\mathbf{y})+h(\mathbf{x}\wedge\mathbf{y})\geq h(\mathbf{x})+h(\mathbf{y})$. *h is* ***log-supermodular (log-spm)***[8] *if it is non-negative,*[9] *and, for all* $\mathbf{x},\mathbf{y}\in X$, $h(\mathbf{x}\vee\mathbf{y})\cdot h(\mathbf{x}\wedge\mathbf{y})\geq h(\mathbf{x})\cdot h(\mathbf{y})$.

Log-supermodularity arises in many contexts in economics. First, supermodularity and log-spm are both sufficient conditions for comparative statics predictions (Topkis (1978); Milgrom and Shannon (1994)). Further, observe that sums of supermodular functions are supermodular, and products of log-spm functions are log-spm; thus log-spm is a natural property to study in multiplicatively separable problems. Now consider some examples where economic primitives are log-spm. A demand function $D(P,t)$ is log-spm if and only if the price elasticity, $e(P)=P\cdot D_P(P,t)/D(P,t)$, is nondecreasing in t. A marginal utility function $U'(w+s)$ is log-spm in (w,s) (where w often represents initial wealth and s represents the return to a risky asset) if and only if the utility satisfies decreasing absolute risk aversion (DARA). A parameterized distribution $F(s;q)$ has a *hazard rate* which is nondecreasing in q if $1-F(s;q)$ is log-spm, that is, if $f(s;q)/[1-F(s;q)]$ is nonincreasing in q. In their study of auctions, Milgrom and Weber (1982) define a related property, *affiliation*, and show that a vector of random variables vector **s** is *affiliated* if and only if the joint density, $f(\mathbf{s})$, is log-spm (almost everywhere).

Finally, log-supermodularity is closely related to a property of probability distributions known as *monotone likelihood ratio order* (MLR). When the support of $F(s;q)$, denoted supp[F],[10] is constant in q, and F has a density f, then the MLR requires that the likelihood ratio $f(s;q_H)/f(s;q_L)$ is nondecreasing in s for all $q_H>q_L$, that is, f is log-spm.[11] However, we wish to define this property for a larger class of distributions, perhaps distributions with varying support and without densities (with respect to Lebesgue measure) everywhere.[12]

Definition 3 *Let* $C(s;q_H,q_L)=\frac{1}{2}\big(F(s;q_L)+F(s;q_H)\big)$. *The parameter q indexes the distribution* $F(s;q)$ *according to the* ***Monotone Likelihood Ratio Order*** *(MLR) if, for all* $q_H>q_L$, $dF(s;q)/dC(s;q_H,q_L)$ *is log-spm for C-almost all* $s_H>s_L$ *in* $\Re^2$.

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8 Karlin and Rinott (1980) referred to this property as multivariate total positivity of order 2.

9 We include non-negativity in the definition to avoid restating the qualification throughout the text. Although supermodularity can be checked pairwise (see Topkis, 1978), Lorentz (1953) and Perlman and Olkin (1980) establish that the pairwise characterization of log-spm requires additional assumptions, such as strict positivity (at least throughout order intervals).

10 Formally, supp[F] $\equiv\{s\,|\,F(s+e)-F(s-e)>0\quad\forall e>0\}$.

11 Note that the MLR implies First Order Stochastic Dominance (FOSD) (but not the reverse): the MLR requires that for any two-point set K, the distribution conditional on K satisfies FOSD (this is further discussed in Athey (1995)).

12 In particular, absolute continuity of $F(\cdot;q_H)$ with respect to $F(\cdot;q_L)$ on the intersection of their supports is a consequence of the definition, not prerequisite for comparability.

Note that we have not restricted F to be a *probability* distribution.

This paper further makes use of an ordering over sets when studying sets of optimizers of a function and how they change with exogenous parameters, as well as in applications where parameters or choice variables change the domain of integration for an expectation. The order is known as Veinott's *strong set order*, defined as follows:

Definition 4 *A set A is greater than a set B in the* ***strong set order*** *(SSO), written A≥B, if, for any a in A and any b in B,* $a \vee b \in A$ *and* $a \wedge b \in B$*. A set-valued function A(t) is nondecreasing in the strong set order (SSO) if for any* $t_H > t_L$*,* $A(t_H) \geq A(t_L)$*. A set A is a* ***sublattice*** *if and only if A≥A.*

If a set $A(t)$ is nondecreasing in t in the strong set order, then the lowest and highest elements of this set are nondecreasing.[13] Any set $[a_1,b_1]\times[a_2,b_2]$ is a sublattice of $\Re^2$, and further such a set is increasing (SSO) in a_i and b_i, i=1,2. These properties arise both in the study of comparative statics (see Topkis 1978) and in our analysis of Section 3.

3. Comparative Statics with Log-Supermodular Payoffs and Densities

This section considers objective functions of the form $U(\mathbf{x},q)\equiv\int u(\mathbf{x},\mathbf{s})\,f(\mathbf{s},q)dm(\mathbf{s})$. Throughout the paper, we assume that u and f are bounded, measurable functions, bold variables are real vectors of finite dimension, and m is a non-negative s-finite product measure. We thus allow for the possibility that $\int_S f dm$ is a probability measure, but do not require it.

In this section, we will focus on problems where one or both of the primitives, u and f, are assumed to be non-negative and log-spm. Applications to investment problems and pricing games are provided in Sections 3.1 and 3.2, while problems with conditional expectations are considered in Section 3.3. The question we seek to ask in this context is a question about monotone comparative statics: that is, when does the following condition hold?

$$\mathbf{x}^*(q)= \arg\max_{\mathbf{x}\in B} U(\mathbf{x},q)\equiv\int u(\mathbf{x},\mathbf{s})\,f(\mathbf{s},q)dm(\mathbf{s}) \text{ is nondecreasing in } q. \qquad \text{(MCSa)}$$

By Milgrom and Shannon (1994), (MCSa) holds and further x^* is nondecreasing in B, if and only if U is quasi-supermodular in $\mathbf{x}$[14] and satisfies SC2 in ($\mathbf{x}$;q). Since an empty set is always larger and smaller than any other set in the strong set order, we do not state an assumption about the existence of an optimum (following Milgrom and Shannon (1994)). In our context, however, we wish to allow for some generality in the specification of the primitives. Thus, we ask, when does (MCSa) hold for all u in some class of functions? We begin by considering the question for the class of log-spm u. The following result is the first step in the analysis of this question.

Lemma 1 *Suppose u and f are nonnegative, and* $u\cdot f>0$ *for* $\mathbf{s}$ *on a set of positive m-measure. Then (i) (MCSa) holds for all* $u(\mathbf{x},\mathbf{s})$ *log-spm, if and only if (ii)* $U(\mathbf{x},q)$ *is log-spm for all* $u(\mathbf{x},\mathbf{s})$ *log-spm.*

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[13] For more discussion of the strong set order, see Milgrom and Shannon (1994).

[14] If X is a product set, U is quasi-supermodular if and only if it satisfies SC2 in each pair (x_i;x_j).

Proof: If $U(\mathbf{x},q)$ is log-spm, then it must be quasi-supermodular, which Milgrom and Shannon (1994) show implies the comparative statics conclusion. Now suppose that $U(\mathbf{x},q)$ fails to be log-supermodular for some u. Our assumptions imply $U(\mathbf{x},q)>0$. Consider first $\mathbf{x}\in\Re^2$. Then, there exists an $x_{1H} > x_{1L}$, $x_{2H} \geq x_{2L}$ and $q_H \geq q_L$ such that $U(x_{1H},x_{2H},q_H)/U(x_{1L},x_{2H},q_H) < U(x_{1H},x_{2L},q_L)/U(x_{1L},x_{2L},q_L)$. Let $\gamma=U(x_{1L},x_{2L},q_L)/U(x_{1H},x_{2L},q_L)>0$. Then, let $b(x_1) = 1$ if $x_1\neq x_{1H}$, and $b(x_{1H}) = g$. Since log-supermodularity is preserved by multiplication, $v(\mathbf{x},\cdot)\equiv u(\mathbf{x},\cdot)\cdot b(x_1)$ is log-supermodular. But then, $V(x_{1H},x_{2H},q_H)/V(x_{1L},x_{2H},q_H) = \gamma U(x_{1H},x_{2H},q_H)/U(x_{1L},x_{2H},q_H) < 1 = \gamma U(x_{1H},x_{2L},q_H)/U(x_{1L},x_{2L},q_H) = V(x_{1H},x_{2L},q_L)/V(x_{1L},x_{2L},q_L)$. Thus, $V(x_{1H},x_{2L},q_L) = V(x_{1L},x_{2L},q_L)$ while $V(x_{1H},x_{2H},q_H) < V(x_{1L},x_{2H},q_H)$, violating quasi-supermodularity and thus the comparative statics conclusion. The argument can be easily extended to multi-dimensional $\mathbf{x}$.

From Milgrom and Shannon's result, it follows that Lemma 1 (ii) is equivalent to the statement that $U(\mathbf{x},q)$ is SC2 in $(\mathbf{x};q)$ for all $u(\mathbf{x},\mathbf{s})$ log-spm. This is somewhat surprising since log-spm is a much stronger property than SC2. However, the result then motivates the main technical question for this section: when is $U(\mathbf{x},q)$ log-spm, given that one of the primitives is log-spm?

Observe that characterizing log-spm of U is non-trivial, since the function $\ln(\cdot)$ is not a linear function. Thus, it is not immediate that properties that hold for $\ln(u)$ will hold for $\ln(\int u(\mathbf{x},\mathbf{s})f(\mathbf{s},q)dm(\mathbf{s}))$. To address this problem, we introduce a result from the statistics literature, which will be one of the main tools in this paper.

Lemma 2 *(Ahlswede and Daykin, 1979) Consider four nonnegative functions, h_i (i=1,...,4), where $h_i:\Re^n\rightarrow\Re$. Then condition (L2.1) implies (L2.2):*

$$h_1(\mathbf{s})\cdot h_2(\mathbf{s}')\leq h_3(\mathbf{s}\vee\mathbf{s}')\cdot h_4(\mathbf{s}\wedge\mathbf{s}') \text{ for } m\text{-almost all } \mathbf{s},\mathbf{s}'\in\Re^n \qquad \text{(L2.1)}$$

$$\int h_1(\mathbf{s})dm(\mathbf{s})\cdot\int h_2(\mathbf{s})dm(\mathbf{s})\leq\int h_3(\mathbf{s})dm(\mathbf{s})\cdot\int h_4(\mathbf{s})dm(\mathbf{s}). \qquad \text{(L2.2)}$$

Karlin and Rinott (1980) provide a simple proof of this lemma. They further explore a variety of interesting applications in statistics, though they do not consider the problem of comparative statics. While we will use this result in a variety of ways throughout the paper, the most important (and immediate) consequence of Lemma 2 for comparative statics is that log-supermodularity is preserved by integration. To see this, set

$$h_1(\mathbf{s})=g(\mathbf{y},\mathbf{s}),\ h_2(\mathbf{s})=g(\mathbf{y}',\mathbf{s}),\ h_3(\mathbf{s})=g(\mathbf{y}\vee\mathbf{y}',\mathbf{s}), \text{ and } h_4(\mathbf{s})=g(\mathbf{y}\wedge\mathbf{y}',\mathbf{s}).$$

Then (L2.1) states exactly that $g(\mathbf{y},\mathbf{s})$ is log-spm in $(\mathbf{y},\mathbf{s})$, while (L2.2) reduces to the conclusion is that $\int g(\mathbf{y},\mathbf{s})dm(\mathbf{s})$ is log-spm in $\mathbf{y}$. Recall that arbitrary sums of log-spm functions are *not* log-spm, which makes this result somewhat surprising. But notice that Lemma 2 does not apply to arbitrary sums, only to sums of the form $g(\mathbf{y},\mathbf{s}^1) + g(\mathbf{y},\mathbf{s}^2)$, when g is log-spm in *all* arguments.

The preservation of log-spm under integration is especially useful for analyzing expected values of payoff functions. Since arbitrary products of log-spm functions are log-spm, a sufficient condition for $\int_{\mathbf{s}} u(\mathbf{x},\theta,\mathbf{s})f(\mathbf{s},\mathbf{x},\theta)dm(\mathbf{s})$ to be log-spm is that $u(\mathbf{x},\theta,\mathbf{s})$ and $f(\mathbf{s},\mathbf{x},\theta)$ are log-spm. Karlin and Rinott give many examples of densities that are log-spm, and thus preserve log-spm of a payoff function.[15]

Despite the fact that log-spm of primitives arises naturally in many economic problems, it is still somewhat restrictive. Thus, we consider the issue of necessary conditions for log-spm to hold. Before stating the necessity theorem, we introduce a definition that will allow us to state concisely theorems about stochastic objective functions throughout the paper. All of the results in this paper concern problems of the form $\int_{\mathbf{s}} u(\mathbf{x},\mathbf{s}) f(\mathbf{s},\boldsymbol{\theta}) dm(\mathbf{s})$, and in such problems, we wish to find *pairs* of hypotheses about *u* and *f* which guarantee that a property of $U(\mathbf{x},\boldsymbol{\theta})$ will hold. We thus look for what we call a minimal pair of sufficient conditions, defined as follows:

Definition 5 *Two hypotheses H-A and H-B are a* ***minimal pair of sufficient conditions (MPSC)*** *for the conclusion C if: (i) Given H-B, H-A is equivalent to C. (ii) Given H-A, H-B is equivalent to C.*

This definition introduces a phrase to describe the idea that we are looking for a pair of sufficient conditions that cannot be weakened without placing further structure on the problem. In some contexts, *H-A* will be given (such as an assumption on *u*), and we will search for the weakest hypothesis *H-B* (such as an assumption on *f*) which preserves the conclusion; in other problems, the roles of *H-A* and *H-B* will be reversed. Though there is no logical requirement that part (ii) of Definition 5 will hold whenever (i) does, the main results of this paper satisfy all three parts of the definition. This definition is used to state the following theorem.

Theorem 1 *Suppose* $u\!:\Re^l \times \Re^n \to \Re_+$ *and* $f:\Re^n \times \Re^m \to \Re_+$, *where* $n \geq 2$ *implies* $l,m \geq 2$. *Then (A)* $u(\mathbf{x},\mathbf{s})$ *is log-spm a.e.-m; and (B)* $f(\mathbf{s},\boldsymbol{\theta})$ *is log-spm a.e.-m; are a MPSC for (C)* $\int_{\mathbf{s}} u(\mathbf{x},\mathbf{s}) f(\mathbf{s},\boldsymbol{\theta}) dm(\mathbf{s})$ *is log-spm in* $(\mathbf{x},\boldsymbol{\theta})$.

Remark *Theorem 1 requires that* **x** *has at least two components (l=2) if n³2, so that the class of u's is sufficiently rich to make log-supermodularity of f in* **s** *a necessary condition. However, even in cases where n³2 and l=1, log-supermodularity of f in* (s_i,q) *is necessary for the conclusion (T1-C) to hold whenever (T1-A) does.*

Theorem 1, which is proved formally in the appendix, states that, not only do (A) and (B) imply the conclusion, but neither restriction can be relaxed without placing additional restrictions on the other primitive. While in Theorem 1, the conditions on *u* and *f* are the same (log-spm), in our subsequent theorems we will pair different conditions, such as single crossing and log-spm, for different conclusions. Together with Lemma 1, this result can be used to provide necessary and sufficient conditions for comparative statics conclusions. Before proceeding to that result, we develop the proof of Theorem 1, and identify its limitations.

The proof of Theorem 1 can be understood with reference to the stochastic dominance

15 For example, symmetric, positively correlated normal or absolute normal random vectors have a log-supermodular density (but arbitrary positively correlated normal random vectors do not; Karlin and Rinott (1980) give restrictions on the covariance matrix which suffice); and multivariate logistic, *F*, and gamma distributions have log-supermodular densities.

literature, and more generally Athey (1995), which exploits a "convex cone" approach to characterizing properties of stochastic objective functions. Athey (1995) shows that in stochastic problems, if one wishes to establishes that a property P holds for $U(\mathbf{x},q)$ for all u in a given class U, it is often necessary and sufficient to check that P holds for all u in the set of extreme points of U. The extreme points can be thought of as test functions; for example, when U is the set of nondecreasing functions, the set of test functions is the set of indicator functions for sets $\mathbf{1}_A(\mathbf{s})$ that are nondecreasing in $\mathbf{s}$. Athey (1995) shows that this approach is the right one when P is a property preserved by convex combinations, *unlike* log-supermodularity.

Despite the fact that the "convex cone" approach does not apply directly here, an analogous intuition can still be developed. Consider the hypothesis that the set of "test functions" for log-supermodular functions is the set of indicator functions of the form $\mathbf{1}_{B_e(\mathbf{x})}$, where $B_e(\mathbf{x})$ is defined to be a cube of length e around the point $\mathbf{x}$. This set of test functions will certainly yield T1-B as a necessary condition for T1-C: only if $f(\mathbf{s},\theta)$ is log-supermodular almost everywhere-μ will $\int_{\mathbf{s}}\mathbf{1}_{B_e(\mathbf{x})}(\mathbf{s})f(\mathbf{s},\theta)dm(\mathbf{s})$ be log-supermodular in $(\mathbf{x};\theta)$. But, it remains to establish that this set of test functions satisfies T1-A. The following lemma can be used:

Lemma 3 *The indicator function $\mathbf{1}_{A(t)}(\mathbf{s})$ is log-spm in $(\mathbf{s},t)$ if and only if $A(t)$ is nondecreasing (strong set order). Further, $\mathbf{1}_A(\mathbf{s})$ is log-spm in $\mathbf{s}$ if and only if A is a sublattice.*

Proof: Simply verify the inequalities and check the definitions. $\mathbf{1}_{A(t)}(\mathbf{x})$ is log-spm in $(\mathbf{s},t)$ if and only if: $\mathbf{1}_{A(t_H)}(\mathbf{s}\vee\mathbf{s}')\cdot\mathbf{1}_{A(t_L)}(\mathbf{s}\wedge\mathbf{s}')\geq\mathbf{1}_{A(t_H)}(\mathbf{s})\cdot\mathbf{1}_{A(t_L)}(\mathbf{s}')$. The definitions imply that if the right-hand side equals one, the left-hand side must equal one as well. Likewise, $\mathbf{1}_A(\mathbf{s})$ is log-spm in $\mathbf{s}$ if and only if: $\mathbf{1}_A(\mathbf{s}\vee\mathbf{s}')\cdot\mathbf{1}_A(\mathbf{s}\wedge\mathbf{s}')\geq\mathbf{1}_A(\mathbf{s})\cdot\mathbf{1}_A(\mathbf{s}')$. Again, this corresponds exactly to the definition.

It is straightforward to verify that sets of the form $\times_i[a_i,b_i]$ are sublattices, as desired. Further, such sets are nondecreasing (strong set order) in each endpoint. Lemma 3 leads to an important role for the strong set order and sublattices in the analysis of stochastic optimization problems. Lemma 3 will also be useful in many applications, for example (as the pricing game of Section 3.2) when the choice variables or parameters affect the domain of integration.

The analogy to the "test functions" approach, and thus the proof of Theorem 1, can be made complete with the following lemma.

Lemma 4 (Test functions for log-spm problems) *Define the set*
$T(b)\equiv\{u:\exists\ \mathbf{x}$ and $0<e<b$ such that $u(\mathbf{x},\mathbf{s})=\mathbf{1}_{B_e(\mathbf{x})}(\mathbf{s})\}$.
Then $U(\mathbf{x},\theta)=\int_{\mathbf{s}}u(\mathbf{x},\mathbf{s})f(\mathbf{s},\theta)dm(\mathbf{s})$ is log-spm in $(\mathbf{x},\theta)$ whenever $u(\mathbf{x},\mathbf{s})$ is log-spm a.e.-m., if and only if, for some $b>0$, $U(\mathbf{x},\theta)$ is log-spm whenever $u\in T(b)$.

Lemma 4 establishes the limits of Theorem 1 by identifying which additional regularity properties of u will change the conclusion of Theorem 1. For example, the elements of $T(b)$ are clearly *not* monotonic, and thus Theorem 1 does not hold under the additional assumption that u is nondecreasing. In contrast, we can clearly approximate the elements of $T(b)$ with smooth functions, so smoothness restrictions will not alter the conclusion of Theorem 1.

Together, Lemma 1, Theorem 1, and Lemma 3 can be used to establish the first main comparative statics theorem of the paper, which states necessary and sufficient conditions for comparative statics in problems with log-supermodular payoff functions.

Corollary 1.1 (Comparative Statics with Log-Supermodular Primitives) *Consider functions* $w:\Re^l\times\Re^m\times\Re^n\to\Re_+$, $f:\Re^n\times\Re^m\to\Re_+$, $A^j:\Re\to 2^{\Re^n}$, $j=1,..,J$. *Let* $\mathbf{A}(\tau)=\cap_{j=1,..,J}A^j(t_j)$. *Let* $\mathbf{x}^*(\theta,\tau,B)$ $=\text{argmax}_{x\in B}W(\mathbf{x},\theta,\tau)\equiv\int_{\mathbf{s}\in\mathbf{A}(\tau)}w(\mathbf{x},\theta,\mathbf{s})\,f(\mathbf{s};\theta)dm(\mathbf{s})$.

(1) (Necessary and sufficient conditions) *Suppose* $\mathbf{A}(\tau)$ *is a sublattice,* $\theta\in\Re$, *and l³2 if n³2. Then* $\mathbf{x}^*(\theta,\tau,B)$ *is nondecreasing in* θ *for all w log-spm,* if and only if *f is log-spm a.e.-m on* $\mathbf{A}(\tau)$.
(2) (General sufficiency) $\mathbf{x}^*(\theta,\tau,B)$ *is nondecreasing in* (θ,τ,B) *if w and f are log-supermodular, and, for each j,* $A^j(t_j)$ *is nondecreasing (strong set order) in* t_j.

Proof: (1) Necessity: by Lemma 1, the comparative statics conclusion is equivalent to log-supermodularity of W in $(\mathbf{x},\theta)$. Then we can apply Theorem 1. Sufficiency follows by: (2) Since products of log-spm functions are log-spm, the function $w(\mathbf{x},\theta,\mathbf{s})\,f(\mathbf{s};\theta)\cdot\mathbf{1}_{A^1(t_1)}(\mathbf{s})\cdots\mathbf{1}_{A^J(t_J)}(\mathbf{s})$ is log-spm in $(\mathbf{s},\mathbf{x},\theta,\tau)$ by Lemma 3.

Note that the objective function analyzed in Corollary 1.1 is not a conditional expectation, since we have not divided by the probability that $\mathbf{s}\in\mathbf{A}(\tau)$. Further, observe that (1) is not stated in terms of a minimal pair of sufficient conditions. The reason is that Lemma 1 only yields necessary conditions for comparative statics if we quantify over all payoffs $u(\mathbf{x},\mathbf{s})$, where u contains the choice variables. In general, log-spm is sufficient, but not necessary, for MCSa.

While Corollary 1.1 is straightforward given the building blocks outlined above, it shows that problems a structure commonly encountered in economic theory can be analyzed by checking a few simple conditions. The following applications illustrate the theorem.

3.1. Applications to Investment Under Uncertainty

An immediate implication of Corollary 1.1 is that a ratio orderings over distributions, commonly used in the literature on investment under uncertainty, can be easily compared: loosely, integrating a function makes it more likely to be log-spm. If a distribution satisfies the MLR order (that is, the density is log-spm), then the corresponding cumulative distribution function $F(s;q)$ will also be log-spm,[16] which can be shown to be stronger than First Order Stochastic Dominance. This will in turn imply that $\int_{-\infty}^{a}F(s;q)ds$ is log-spm, which is stronger than Second Order Stochastic Dominance if q does not change the mean of s.

More generally, we can consider any ratio ordering which specifies that $g(s)/h(s)$ is nondecreasing in s for $g,h>0$. The easiest way to fit this into our framework is to define $u(x,s)$ on

[16] See Eeckhoudt and Gollier (1995) who term this the monotone probability ratio (MPR) order, and show that an MPR shift is sufficient for a risk-averse investor to increase his portfolio allocation.

$\{0,1\}\times\Re$, and let $u(x,s) = g(s)$ if x=1, and let $u(x,s) = h(s)$ if x=0. Then, if $f(s;q)$ is log-spm and $g(s)/h(s)$ is nondecreasing in s, $E[g(s)|q]/E[h(s)|q]$ must be nondecreasing in q. Further, Theorem 1 shows that log-spm of f is the weakest condition which preserves *every* ratio ordering.

An example from the economics of uncertainty is the Arrow-Pratt coefficient of risk aversion, $R(w;u)\equiv -u''(w)/u'(w)$. Suppose that an agent has decreasing absolute risk aversion (DARA) and faces a risk s with distribution $F(s;q)$. Let $U(w;q) = \int u(w+s)f(s;q)ds$, and observe that $u'(w+s)$ is log-spm if and only if $R(w;u)$ is nonincreasing in w (DARA). Theorem 1 then implies that in response to an MLR shift in the risky asset s, the agent will be less risk averse when considering a new, independent risk. But, observe that the fact that w and s enter u additively precludes the test functions of Lemma 4, so necessity cannot be obtained using Theorem 1 for risk averse agents.

A further consequence of Theorem 1 is that orderings over risk aversion and related properties are preserved with respect to background risks. Let $U(w) = \int u(w+s)f(s)ds$. If $R(w;u)$ is decreasing, it then follows from Theorem 1 that $U'(w+t)$ will be decreasing as well, implying that $R(w;U)$ is decreasing. Similar analyses apply to other ratio orderings, such as Kimball's (1990) decreasing prudence condition,[17] where prudence is defined by $-u'''(w)/u''(w)$.[18] More generally, we can consider orderings of utilities over risk aversion (not just those induced by a shift in wealth). Let q parameterize the agent's risk aversion directly, and let $U(w;q) = \int u(w+s;q)f(s)ds$. Then if $R(w;u(\cdot;q))$ is nonincreasing in w and q, U inherits these properties, so that $R(w;U(\cdot;q))$ is nonincreasing in q and w. Other ratio orderings can be similarly analyzed.

Finally, Theorem 1 allows us to consider more general background risks, those which might be statistically dependent on the risk of primary interest. Let z be the "primary" risk, and let $\mathbf{s}$ be a vector of assets in which the agent also has a position, represented by the vector of positive portfolio weights α. The agent's utility is given by $U(z,q) = \int u(z+\alpha\cdot\mathbf{s};q)f(\mathbf{s}|z)d\mathbf{s}$. Suppose that the risks are affiliated, that is, $f(\mathbf{s},z)$ is log-spm. But, applying the multivariate Theorem 1, the agent's risk aversion with respect to z (that is, $R(z;U,q)$) will be nonincreasing in q, if u satisfies DARA and q decreases risk aversion. An alternative, but distinct sufficient condition requires that the conditional density of $y=\alpha\cdot\mathbf{s}$ given z, $g(y|z)$, is log-spm.

This analysis thus provides several extensions to the existing literature. Jewitt (1987) showed that the "more risk averse" ordering is preserved by a MLR shift in the distribution, while Pratt (1988) established that risk aversion orderings are preserved by expectations. Eeckhoudt, Gollier, and Schlesinger (1996) provide additional conditions on distributions under which a FOSD shift in a background risk decreases risk aversion for a DARA investor. The above analysis generalizes

[17] Kimball (1990) shows that agents who are more prudent will engage in a greater amount of precautionary

[18] It is interesting to observe that since a smooth function $g(x+y)$ is supermodular if and only if it is convex, these results can also be proved using the fact that log-convexity is preserved under expectations (Marshall and Olkin, 1979); but, the correspondence between convexity and supermodularity breaks down with more than two variables.

these results by showing that all of the results extend to other ratio orderings, such as prudence, and further risk aversion orderings are preserved by affiliated background risks for DARA investors.

3.2. Log-Supermodular Games of Incomplete Information

This section establishes sufficient conditions for a class of games of incomplete information to have a PSNE in nondecreasing strategies. Consider a game of incomplete information between many players, each of whom has private information about her own type, t_i, and chooses a strategy $c_i(t_i)$. Vives (1990) observed that the theory of supermodular games (Topkis, 1979) can be applied to games of incomplete information. He showed that if each player's payoff $v_i(\mathbf{x})$ is supermodular in the realizations of actions $\mathbf{x}$, the game has strategic complementarities in strategies $c_j(t_j)$ (that is, a pointwise increase in an opponent's strategy leads to a pointwise increase in a player's best response). This in turn implies that a PSNE exists.

In contrast, if we consider the same problem but assume that $v_i(\mathbf{x})$ is log-spm, it is no longer true that the game has strategic complementarities in stategies in the sense just described. However, we can instead take a different approach: Athey (1997) shows that a PSNE exists in games of incomplete information where an individual player's strategy is nondecreasing in *her own type* (i.e. $c_i(t_i^H)\geq c_i(t_i^L)$), whenever all of her opponents use nondecreasing strategies.

Formally, suppose that $h(\mathbf{t})$ is the joint density over types. Let player i's utility be given by $v_i(\mathbf{x},\mathbf{t})$ (so that opponents' types might influence payoffs both directly and indirectly, through the opponents' choices of actions). Thus, a player's payoff given a realization of types can be written $u_i(x_i,\mathbf{t})\equiv v_i(x_i,\boldsymbol{\chi}_{-i}(\mathbf{t}_{-i}),\mathbf{t})$. Taking the expectation over opponent types yields the following expression for expected payoffs: $U_i(x_i,t_i)\equiv\int_{\mathbf{s}_{-i}} u_i(x_i,\mathbf{t})h_i(\mathbf{t}_{-i}|t_i)d\mathbf{t}_{-i}$. The following result gives sufficient conditions for a player's best response to nondecreasing strategies to be nondecreasing.

Proposition 1 *Let $h:\Re^n\rightarrow\Re_+$, where $h(\mathbf{t})$ is a probability density. Assume that h has a fixed, convex support. For i=1,..,n, let $h_i(\mathbf{t}_{-i}|t_i)$ be the conditional density of $\mathbf{t}_{-i}$ given t_i, and define payoffs as above. Then (i) for all i, $c_i(t_i) = \arg\max_{x_i} U(x_i,t_i)$ is nondecreasing (strong set order) in t_i for all $c_j(t_j)$ nondecreasing for $j\neq i$, and v_i log-spm, if and only if (ii) $h(\mathbf{t})$ is log-spm almost everywhere-Lebesgue on the support of* $\mathbf{t}$.

Proof: Sufficiency follows from Lemma 2 and Milgrom and Shannon (1994). Following the proof of Theorem 1, it is possible to show that $U_i(x_i,t_i)$ is log-supermodular for all $u_i(x_i,\mathbf{t})$ log-supermodular, only if, for all $\mathbf{t}_{-i}^H>\mathbf{t}_{-i}^L$ and all $t_i^H>t_i^L$, $h_i(\mathbf{t}_{-i}^H|t_i^H)h_i(\mathbf{t}_{-i}^L|t_i^L)\geq h_i(\mathbf{t}_{-i}^L|t_i^H)h_i(\mathbf{t}_{-i}^H|t_i^L)$. But, since for a positive function, log-spm can be checked pointwise, this condition holds for all i if and only if h is log-spm. Apply Lemma 1.

Spulber (1995) recently analyzed how asymmetric information about a firms' cost parameters alters the results of a Bertrand pricing model, showing that there exists an equilibrium where prices are increasing in costs, and further firms price above marginal cost and have positive expected profits. Spulber's model assumes that costs are independently and identically distributed, and that values are private; Proposition 1 easily generalizes his result to asymmetric, affiliated signals, and to imperfect substitutes. Let $v_i(\mathbf{x},\mathbf{t}) = (x_i - t_i)\cdot D_i(\mathbf{x})$, where $\mathbf{x}$ is the vector of prices, $\mathbf{t}$ is the vector of marginal costs, and $D_i(\mathbf{x})$ gives demand to firm i when prices are $\mathbf{x}$.

By Theorem 1, the expected demand function is log-spm if the signals are affiliated, each opponent uses a nondecreasing strategy, and $D_i(\mathbf{x})$ is log-spm. The interpretation of the latter condition is that the elasticity of demand is a non-increasing function of the other firms' prices. Demand functions which satisfy these criteria include logit, CES, transcendental logarithmic, and a set of linear demand functions (see Milgrom and Roberts (1990b) and Topkis (1979)). Further, when the goods are perfect substitutes, expected demand is also log-spm. To see this, note that when each opponent uses a nondecreasing strategy, expected demand is given by

$$D_1(x_1)\int_{\mathbf{t}_{-1}} \mathbf{1}_{c_2(t_2)>x_1}(t_2)\cdots\mathbf{1}_{c_n(t_n)>x_1}(t_2)\, h(\mathbf{t}_{-1}|t_1)d\mathbf{t}_{-1}$$

and the set $\{t_j : c_j(t_j) > x_1\}$ is nondecreasing (strong set order) in x_1 when c_j is nondecreasing. Then, by Lemma 3 and Corollary 2.1, expected demand must be log-spm when the density is.

Thus, a PSNE exists in nondecreasing pricing strategies whenever marginal cost parameters are affiliated and demand is log-spm and continuous, or in the case of perfect substitutes.

3.3. *Conditional Stochastic Monotonicity and Comparative Statics*

Corollary 2.1 applies to stochastic problems where the domain of integration is restricted, but not to conditional expectations. Conditional expectations of multivariate payoff functions arise in a number of economic applications, such as the "mineral rights auction" (Milgrom and Weber, 1982). This section studies comparative statics when the agent's objective takes the form

$$U(x,q|A(t)) \equiv \int_A u(x,\mathbf{s})\, f(\mathbf{s}|q, A(t))dm(\mathbf{s}) = \int_{A(t)} u(x,\mathbf{s}) \frac{f(\mathbf{s},q)}{\int_{A(t)} f(\mathbf{s},q)dm(\mathbf{s})} dm(\mathbf{s})\,.$$

The comparative statics condition of interest in this problem is

$$x^*(q,t,B) = \arg\max_{x\in B} U(x,q|A(t)) \text{ is nondecreasing in } (q,t). \qquad \text{(MCSb)}$$

Before proceeding, we need another definition: we say that $u(x,\mathbf{s})$ satisfies *nondecreasing differences* in $(x;\mathbf{s})$ if $u(x^H,\mathbf{s}) - u(x^L,\mathbf{s})$ is nondecreasing in $\mathbf{s}$ for all $x^H \geq x^L$. In terms of the vector $\mathbf{s}$, this assumption is clearly weaker than log-supermodularity of u in $\mathbf{s}$, since no assumptions on the interactions between the components of $\mathbf{s}$ are imposed; however, nondecreasing differences is neither weaker nor stronger than log-spm, unless u satisfies additional monotonicity restrictions.

Our focus on supermodularity of $U(x,q|A(t))$ in the study of comparative statics in problems for this class of payoff functions is motivated by the following result (Athey, 1995), which is analogous to Lemma 1: for a given τ, $U(x,q|A(t))$ satisfies SC2 in (x,q) for all u which satisfy nondecreasing differences in $(x;\mathbf{s})$, if and only if $U(x,q|A(t))$ in $(x;q)$ is supermodular for all u which satisfy nondecreasing differences in $(x;\mathbf{s})$.

Of course, $U(x,q|A(t))$ is supermodular in (x,q) if and only if $U(x^H,q|A(t)) - U(x^L,q|A(t))$ is nondecreasing in q for all $x^H > x^L$, and thus the problem reduces to checking that $G(q|A(t)) \equiv \int_A g(\mathbf{s}) f(\mathbf{s},q|A(t))dm(\mathbf{s})$ is nondecreasing in q and t for all g nondecreasing. The following theorem applies to this problem:

Theorem 2 *Consider $f:\Re^{n+1} \to \Re_+$. Then (A) $g:S \to \Re$ is nondecreasing a.e.-m; and (B) $f(\mathbf{s};q)$ is log-spm a.e.-m; are a MPSC for (C) $G(q|A(t))$ is nondecreasing in t and q for all $A(t)$ nondecreasing in the strong set order.*

Theorem 2 is a generalization of a result which has received attention in both statistics and economics (Sarkar (1968), Whitt (1982), Milgrom and Weber (1982)).[19] To see the proof of sufficiency using Lemma 2, consider the case where g is nonnegative. Consider $t_H > t_L$ and $q_H > q_L$, and define the following functions:

$$h_1(\mathbf{s}) = g(\mathbf{s}) \cdot \mathbf{1}_{\mathbf{s}\in A(t_L)}(\mathbf{s}) \cdot f(\mathbf{s};q_L),\; h_2(\mathbf{s}) = \mathbf{1}_{\mathbf{s}\in A(t_H)}(\mathbf{s}) \cdot f(\mathbf{s};q_H),$$

$$h_3(\mathbf{s}) = g(\mathbf{s}) \cdot \mathbf{1}_{\mathbf{s}\in A(t_H)}(\mathbf{s}) \cdot f(\mathbf{s};q_H), \text{ and } h_4(\mathbf{s}) = \mathbf{1}_{\mathbf{s}\in A(t_L)}(\mathbf{s}) \cdot f(\mathbf{s};q_L).$$

It is straightforward to verify under our assumptions that $h_1(\mathbf{s}) \cdot h_2(\mathbf{s}') \le h_3(\mathbf{s} \vee \mathbf{s}') \cdot h_4(\mathbf{s} \wedge \mathbf{s}')$ (using Lemma 3 and the fact that log-spm is preserved by multiplication); thus, Lemma 2 gives us

$$\int_{\mathbf{s}\in A(t_L)} g(\mathbf{s}) f(\mathbf{s};q_L)dm(\mathbf{s}) \cdot \int_{\mathbf{s}\in A(t_H)} f(\mathbf{s};q_H)dm(\mathbf{s}) \le \int_{\mathbf{s}\in A(t_H)} g(\mathbf{s}) f(\mathbf{s};q_H)dm(\mathbf{s}) \cdot \int_{\mathbf{s}\in A(t_L)} f(\mathbf{s};q_L)dm(\mathbf{s}).$$

Rearranging the inequalities gives the desired ranking, $G(q_L|A_L) \le G(q_H|A_H)$.

The stochastic monotonicity results can then be exploited to derive necessary and sufficient conditions for comparative statics conclusion in problems of the $U(x,q,t)$. In particular, we have the following corollary:

Corollary 2.1 (Comparative Statics with Conditional Expectations) *Consider any $A(t)$, and suppose $F(\mathbf{s},q)$ is a probability distribution. Then MCSb holds for all u which satisfy nondecreasing differences in $(\mathbf{s};q)$, if and only if $f(\mathbf{s},q)$ is log-supermodular a.e.-m.*

Finally, we examine the limits of the necessity part of Theorem 2. The counter-examples we construct require us to condition on an arbitrary sublattice. If the economic problem places additional structure on the set A, log-spm is no longer necessary. Consider a particular case which arises very commonly in applications (such as auctions): assume there is a single random variable, and let $A(\tau)=1_{\{s<\tau\}}$. In this case, log-supermodularity of the density is not necessary for monotone comparative statics predictions, but instead, we require the *distribution* to be log-spm.

Corollary 2.2 *Consider a probability distribution $F(s;q)$, $s\in\Re$, and restrict attention to $A(\tau)=1_{\{s<\tau\}}$. Then MCSb holds for all $u(x,s)$ supermodular, if and only if $F(s;q)$ is log-spm.*

4. Comparative Statics with Single-Crossing Payoffs and Densities

This section studies single crossing properties in problems where there is a single random

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[19] Sarkar (1968) establishes sufficiency, and Whitt (1982) studies conditions under which $G(q|A)$ is increasing in q for all sublattices A; Milgrom and Weber (1982) study conditions under which $G(q|A)$ is increasing in q and also in a limited class of shifts in A.

variable. Multivariate generalizations of the results are sufficiently restrictive that they are not considered here. However, many problems in the theory of investment under uncertainty, auctions, and signaling games can be fruitfully analyzed with a single random variable.

Problems of the form $U(x,q)\equiv\int u(x,s)\,f(s;q)dm(s)$, where all variables are real numbers, are a special case of those considered in Section 3. This section seeks to relax the assumption that both primitives are log-spm. In particular, we consider the weaker assumption that $u(x,s)$ satisfies SC2 in $(x;s)$. Problems which fit into this framework include mineral rights auction games and investment under uncertainty problems. The problem of interest is

$$x^*(q,B)=\text{argmax}_{x\in B}\, U(x,q)\equiv\int u(x,s)\,f(s;q)dm(s) \text{ is nondecreasing in } q. \qquad \text{(MCSc)}$$

Our results from Section 3 give some initial insight into this problem: they imply that a *necessary* condition for (MCSc) to hold for all u which are SC2 is that $f(s,q)$ is log-spm. a.e.-μ. To see this, observe that since SC2 is weaker than log-spm, if (MCSc) holds for all $u(x,s)$ which satisfy SC2, then (MCSc) must hold for all $u(x,s)$ log-spm. By Corollary 1.1, this implies that $f(s,q)$ is log-spm a.e.-μ. However, since SC2 is weaker than log-spm, our results from Section 3 do not establish that u SC2 and f log-spm are sufficient for (MCSc). To analyze this question, it will be useful to transform the problem by taking a the first difference with respect to x, since $U(x,q)$ and $u(x,s)$ satisfy SC2 if and only if, for all $x_H>x_L$, $U(x_H,q)-U(x_L,q)$ and $u(x_H,s)-u(x_L,s)$ satisfy SC1. Thus, we study conditions under which $G(q)\equiv\int g(s)k(s,q)dm(s)$ satisfies SC1 in q.

Section 4 proceeds as follows. Section 4.1 analyzes SC1 of $G(q)$ and applications which are most easily analyzed from that perspective, such as the portfolio problem. This section develops the main technical ideas of Section 4. Section 4.2 considers (MSCc) directly, and provides additional applications. Section 4.3 considers comparative statics using the Spence-Mirlees single crossing property (SC3). The main results are summarized in Table 1 in the conclusion.

4.1. SC1 in Stochastic Problems

The main result of this section finds the minimal sufficient conditions for our single crossing conclusion.

Theorem 3 *Let K(s;q) be a distribution. (A) g(s) satisfies SC1 in (s) a.e.-m*[20]*; and (B) K(s;q) satisfies MLR; are a MPSC for (C)* $G(q)=\int g(s)dK(s;q)$ *satisfies SC1 in q.*

Extensions to Theorem 3*:*[21] *Theorem 3 also holds if: (i) there exists k and m such that* $K(s;q)=\int_{-\infty}^{s} k(t,q)dm(t)$ *for all q, in which case (B) is equivalent to k is log-spm a.e.-m; (ii) g depends on q directly, under the additional restrictions that g is piecewise continuous in q*

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[20] We will say that $g(s)$ satisfies SC1 almost everywhere-m (a.e.-m) if conditions (a) and (b) of the definition of SC1 hold for almost all (w.r.t. the product measure on $\Re^2$ induced by m) (s_H,s_L) pairs in $\Re^2$ such that $s_H>s_L$.

[21] A version of this theorem which gives minimal sufficient conditions for *strict* single crossing is provided in Athey (1996).

and g is nondecreasing in q (see also Theorem 5 below for another sufficient condition);
(iii) g(s) satisfies only WSC1, provided supp[$K(s;q)$] *is constant in q.*

The theory of the preservation of single crossing properties under uncertainty has a long history in the statistics literature. Karlin and Rubin (1956) establish sufficiency, and Karlin (1968) (pp. 233-237) analyzes necessary conditions,[22] under the absolute continuity assumption of Extension (i) and some additional regularity conditions (including the assumption that Θ has at least three points).[23] When K is a probability distribution, the ex ante absolute continuity assumption may be undesirable; however, if k represents a utility function, it is the right assumption.

We will outline the proof in the text. The sufficiency proof is surprisingly simple, and our applications will show that it can be easily modified to produce extensions of Theorem 3. In addition, in this section (as we did in Lemma 4 and Section 3.3), we identify the smallest class of functions which is required to generate the counter-examples used to prove necessity; this will be useful for analyzing the tradeoffs between alternative assumptions on primitives. In Section 4.1.1, we illustrate in applications how additional commonly encountered restrictions on g or k can be used to relax (T3-A) and (T3-B).

Consider first sufficiency. Suppose for simplicity that $k(s,q)>0$. Define $l(s)= k(s;q_H)/k(s;q_L)$, which (4.1-B) guarantees is nondecreasing. Notice first that, for a given $q_H > q_L$, $\int g(s)k(s;q_H)dm(s) = \int g(s)\left[k(s;q_H)/k(s;q_L)\right]k(s;q_L)dm(s)$. Let s_0 be a point where g crosses zero. Then:

$$\begin{aligned}\int g(s)k(s;q_H)dm(s) &= \int g(s)l(s)k(s;q_L)dm(s)\\ &= -\int_{-\infty}^{s_0}|g(s)|l(s)k(s;q_L)dm(s)+\int_{s_0}^{\infty}g(s)l(s)k(s;q_L)dm(s) \qquad (4.1)\\ &\geq -l(s_0)\int_{-\infty}^{s_0}|g(s)|k(s;q_L)dm(s)+l(s_0)\int_{s_0}^{\infty}g(s)k(s;q_L)dm(s)=l(s_0)\int g(s)k(s;q_L)dm(s)\end{aligned}$$

The second equality holds because g satisfies SC1, while the inequality follows since monotonicity of l implies the following condition, which is necessary (4.1) to hold:

$$l(s) \leq l(s_0) \text{ for } s<s_0, \text{ while } l(s) \geq l(s_0) \text{ for } s>s_0. \qquad (4.2)$$

(4.2) requires that the likelihood ratio $l(s)$ satisfies WSC1(s_0). (In Section 4.1.1 below, we will show how fixing the crossing point can allow us to exploit (4.2) as a sufficient condition). If, in addition, $l(s_0)\leq 1$, then $\int g(s)k(s;q_H)dm(s)\leq 0$ implies $\int g(s)k(s;q_L)dm(s)\leq\int g(s)k(s;q_H)dm(s)$; this case arises when $k(s;q)$ is a probability distribution (not a density). In that case, if $l(s)$ is nondecreasing, it must always be less than 1, which is its value at the highest s.

The necessity parts of this theorem can be proved by constructing counterexamples, which

[22] I am grateful to an anonymous referee who directed me to this theorem.

[23] The necessity proof provided in Karlin (1968) is designed to solve a much more general problem (about the preservation of an arbitrary number of sign changes, and thus it requires additional regularity assumptions and a complicated construction.

place all of the weight on the failures. Thus, the proof of Theorem 3 makes use of the following:

Lemma 5 (Test functions for single crossing problems) *Define the following sets:*

$\mathcal{G}(b)\equiv$ $\{g$: $\exists a,b>0$, $b>e,d>0$, *and* $s_L<s_H$ *s.t.* $g(s)=-a$ *for* $s\in(s_L-e,s_L+e)$ *and*
$g(s)=b$ *for* $s\in(s_H-e,s_H+e)$, $|g|<d$ *elsewhere, and g satisfies SC1*$\}$

$\mathcal{K}(b)\equiv$ $\{k$: $\exists a>0$, $b>e>0$ *and* s_0 *s.t.* $k(s)=a$ *for* $s\in(s_0-e,s_0+e)$, *and* $k=0$ *elsewhere*$\}$.

Then Theorem 3 holds if, for any $b>0$, *(4.1-A) is replaced with (4.1A')* $g(s)\in\mathcal{G}(b)$; *Theorem 3 also holds if (4.1-B) is replaced with (4.1B')* $k(s,q)\in\mathcal{K}(b)$.

As in Theorem 1, placing smoothness assumptions on g will not change the conclusion of Theorem 3, but monotonicity or curvature assumptions will. If monotonicity of one of the primitives is assumed (for example, if k is a probability distribution, not a density), then the necessity parts of the theorem break down, as shown in Section 4.2.1.

Now consider how the counter-examples are used. If g fails (T3-A), then there is $s_L < s_H$ such that $g(s_L)\geq 0$, but $g(s_H)<0$, on sets S_L and S_H of positive measure m. But then, $k(s;q)$ can be defined so that $k(s;q_H)$ places all of the weight on high points s_H, while $k(s;q_L)$ places all of the weight on the low points s_L. This function is log-spm, but $G(q)$ fails SC1 since g does. If k fails (T3-B), then there exist two sets of positive measure, S_H and S_L, such that increasing q places more weight on S_L relative to S_H. Then we can construct a $g(s)$ that is negative on S_L, positive on S_H, and close to zero everywhere else.

However, the proof of necessity of (T3-B) involves some additional work. Theorem 3 does not place ex ante restrictions on how supp[$K(s;q)$] moves with q. The restrictions implied when (T3-C) holds whenever (T3-A) does are summarized in the following Lemma.

Lemma 6 *If* $\int g(s)dK(s;q)$ *satisfies SC1 in q whenever g satisfies SC1 in s, (i) for all* $q_H>q_L$, $K(s;q_H)$ *is absolutely continuous with respect to* $K(s;q_L)$ *on* $(\inf_s \text{supp}[K(s;q_H)], \sup_s \text{supp}[K(s;q_L)])$, *and (ii)* supp[$K(s;q)$] *is nondecreasing in the strong set order.*

The following applications show how additional structure present in particular problems can be exploited; they also generalize the result to allow k to cross zero.

4.1.1. Extensions and Applications to Investment Problems

This section uses Theorem 3 to analyze two classic problems, the portfolio investment problem and the decision problem of a risk averse firm. We develop extensions to Theorem 3 which allow us to generalize several comparative statics results previously established only for special functional forms.

Consider first the standard portfolio problem, where an agent with initial wealth w invests x in a risky asset s, and receives payoffs $u((w-x)r+sx)$. The first order conditions are given by

$$\int u'((w-x)r+sx)(s-r)f(s,q)ds\,. \tag{4.3}$$

Notice that $s-r$ satisfies single crossing, and further, the crossing point is fixed at $s_0=r$ for all utility functions. This section identifies how (T3-B) can be weakened using this additional structure. We begin with an additional definition.

Definition 5 *We will say that* $k(s;q)$ *satisfies weak single crossing of ratios about* s_0*, denoted* **WSCR(s_0)** *if (i)* $\hat{l}(s_0)=\lim_{s\to s_0} k(s,q^H)/k(s,q^L)$ *exists, (ii)* supp$[K(s,q)]$ *is nondecreasing in the strong set order, (iii) either* $k\geq 0$ *or k satisfies WSC1(*s_0*), and (iv)* $k(s,q_H)/k(s,q_L) - \hat{l}(s_0)$ *satisfies WSC1(*s_0*) for s where* $k(s;q_L)\cdot k(s;q_H)>0$.

While the WSCR(s_0) condition may appear unfamiliar, it is potentially quite useful. In particular, it allows that the function k could itself cross zero, though we will not exploit that fact until the next subsection. To start, we observe that if k is nonnegative, then $k(s,q)$ satisfies WSCR(s_0) for all s_0 if and only if k is log-spm. Further, WSCR(s_0) can be satisfied if $k(s,q_H)$ and $k(s,q_L)$ cross exactly once, at s_0. If k is a probability distribution and the mean of s does not change with θ, it is known that such a single crossing property of the distribution implies that θ indexes k according to second order stochastic dominance.

To see how the weak single crossing property of ratios can be used to establish single crossing conclusions, recall that the inequality in (4.1) holds whenever (4.2) holds. However, for a fixed s_0 and for $k\geq 0$, (4.2) is equivalent to WSCR(s_0). Thus, if we know that g satisfies WSC1(s_0), (4.1) will hold whenever $k(s,q)$ satisfies WSCR(s_0).

Theorem 4*: Suppose that supp(K) is constant in q and k is nonnegative. Then (A)* $g(s)$ *satisfies WSC1(*s_0*) in (s) a.e.-m, and (B)* $k(s;q)$ *satisfies WSCR(*s_0*) a.e.-m, are a MPSC for (C)* $G(q)=\int g(s)k(s,q)dm(s)$ *satisfies SC1 in q*:

Consider how this result relates to Theorem 3. If we relax (T3-A) to allow for WSC1 around *any* point, we must strengthen (T3-B) so that k is restricted to be log-spm. Consider an application of this result. In the portfolio problem, we can apply Theorem 4 to (4.3), letting $g=s-r$ and $k=u'\cdot f$. Then the optimal portfolio allocation $x^*(q,B,r)$ is nondecreasing in q whenever $f(s,q)$ satisfies WSCR(r). But, if we desire a comparative statics result for all r, log-supermodularity of f will be required. In many problems the relevant range for the risk-free rate may be smaller than the support of the risk-free asset, and thus Theorem 4 has content.

The portfolio problem has been widely studied. However, far fewer results have been obtained for more general investment problems, where potentially risk-averse firms invest in a risky projects $p(x,s)$, or make pricing or quantity decisions under uncertainty about demand. Suppose that an agent's objective is as follows: $\max_{x\in B}\int u(p(x,s),q)f(s,q)dm(s)$, and the solution set is denoted $x^*(q,B)$. Thus, p represents a general return function which depends on the investment amount, x, and the state of the world, s. The agent's utility depends on the returns to the project as well as some exogenous parameter q, which might represent initial wealth or some other factor relating to risk aversion. Notice that in this problem, the crossing point of π is not automatically determined as in the portfolio problem.

When this objective function is differentiable, it suffices to check that the marginal returns to x, denoted $\int u_1(p(x,s),q)p_x(x,s)f(s,q)dm(s)$, satisfy SC1. However, we might also wish to allow for the possibility that the agent faces discrete choices between investments, or that the function p is endogenously determined (so that regularity properties of p cannot be assumed). To analyze this problem, we introduce another extension to Theorem 3, as follows:

Theorem 5 *Suppose that (i) supp(K) is constant in q, (ii) g(s,q) satisfies WSCR(s_0) a.e.-m for all q; and (iii) k(s,q) is log-spm a.e.-m. Then* $G(q)=\int g(s,q)k(s,q)dm(s)$ *satisfies SC1 in q.*

Proof: Consider the case where g crosses 0 (otherwise, the expectation is always non-negative) and g=0 only at $s=s_0$, and k>0; the other cases can be handled in a manner similar to the proof of Theorem 3. Then, we extend (4.1) as follows:

$$\int g(s;q_H)k(s;q_H)dm(s)$$
$$\geq \lim_{s\to s_0}\frac{g(s;q_H)k(s;q_H)}{g(s;q_L)k(s;q_L)}\left[-\int_{-\infty}^{s_0}|g(s;q_L)|k(s;q_L)dm(s)+\int_{s_0}^{\infty}g(s;q_L)k(s;q_L)dm(s)\right]$$
$$=\lim_{s\to s_0}\frac{g(s;q_H)k(s;q_H)}{g(s;q_L)k(s;q_L)}\int g(s;q_L)k(s;q_L)dm(s)$$

The inequality follows, as in Theorem 3, because g satisfies WSC1(s_0), and $g\cdot k$ satisfies WSCR(s_0) since g and k do.

Proposition 2 *Consider the problem* $\max_{x\in B}\int u(p(x,s),q)f(s,q)dm(s)$. *Assume that u(y,q) is increasing and differentiable*[24] *in y and p(x,s) is nondecreasing in s. Then:*

(A) p(x,s) satisfies SC2 in (x;s) a.e.-m, and (B) $u_1(y,q)f(s,q)$ *is log-spm. in (s,y,q) a.e.-m,*[25] *are a MPSC for the conclusion (C) x*(q,B) is nondecreasing in q.*

Proof: (A) and (B) imply (C): If π is differentiable in x and B is a convex set, we can analyze whether $\int u_1(p(x,s),q)p_x(x,s)f(s,q)dm(s)$ satisfies SC1. We can let $g=\pi_x$, and let $k=u_1\cdot f$, and apply Theorem 3. Now consider the investor's choice between two values of x, $x_H > x_L$. Then, let $g(s,q) = u(p(x_H,s),q)-u(p(x_L,s),q)$, and let $k(s;q) = f(s;q)$. First, observe that SC2 is preserved under monotone transformations, so that if u is nondecreasing in its first argument, then by (A), $u(p(x,s),q)$ must satisfy SC2 in $(x;s)$, and $g(s,q)$ satisfies SC1 in s. Now, consider conditions under which $g(s,q)$ satisfies WSCR(s_0). Let s_0 be the crossing point of p_x. First restrict attention to $s\geq s_0$, where $p(x_H,s)\geq p(x_L,s)$. Define $h(a,b,q)=\int_{y=a}^{b}u_1(y,q)dy$, and note that h is log-spm in (a,b,q) for all $a<b$, by Corollary 2.1 and (B). This in turn implies that $g(s,q)=h(p(x_L,s),p(x_H,s),q)$ is log-spm in (s,q) on $s\geq s_0$ since p is nondecreasing in s, and thus $g(s;q_H)/g(s;q_L)$ is nondecreasing in s on $s\geq s_0$. On the other hand, if $s<s_0$, $p(x_H,s)\leq p(x_L,s)$, and $g(s,q)= -h(p(x_H,s),p(x_L,s),q)$. Then, $g(s;q_H)/g(s;q_L)$ is nondecreasing in s on $s<s_0$ since h is log-spm, and WSCR(s_0) holds. Then apply Theorem 5. Necessity follows by Theorem 3 for the case where π is differentiable; the proof is omitted for the more general case.

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[24] This is not essential but it simplifies the proof.

[25] If u is not everywhere differentiable in its first argument, the corresponding hypothesis can be stated as follows: $[u(y_H,q)-u(y_L,q)]\cdot f(s;q)$ is log-supermodular in (y_L,y_H,q,s) for all $y_L < y_H$.

Hypothesis (B) is satisfied if (i) q decreases the investor's absolute risk aversion, and (ii) q generates an MLR shift in F. This result provides a general statement of two basic results in the theory of investment under uncertainty, illustrating that log-supermodularity is behind the seemingly unrelated conditions on the distribution and the investor's risk aversion. The proof of Proposition 2 further highlights an application of the tools from Section 3.

The analysis of this section establishes several generalizations of the existing investment literature. This literature typically considers the problem where the objective is differentiable and strictly quasi-concave. Landsberger and Meilijson (1990) shows that the MLR is sufficient for comparative statics in the portfolio problem, and Ormiston and Schlee (1993) show that arbitrary comparative statics results are preserved under the MLR, and suggest a behavioral relationship between risk aversion and the MLR. A few papers consider comparative statics when $p(x,s) = h(x)\cdot s$, as in Sandmo's 1971 classic model of a firm facing demand uncertainty. The focus of Milgrom's (1994) analysis was to show that comparative statics results derived for the portfolio problem also hold for Sandmo's model. Proposition 2 extends the analysis further, highlighting the fact that supermodularity of π, but not multiplicative separability, is critical for the comparative statics conclusions of Proposition 2.[26] Thus, the comparative statics results from portfolio theory and Sandmo's model can be extended to more general models of firm objectives.

4.2. SC2 and Comparative Statics

The main comparative statics Theorem of Section 4 follows immediately from Theorem 3:

Corollary 3.1(Comparative Statics with Single Crossing Payoffs) *(A) u(x,s) satisfies SC2 in (x;s) a.e.-.m. and (B) K(s;q) satisfies MLR; are a MPSC for (C) MSCc holds for all sets B.*[27]

Corollary 3.1 has an interesting interpretation. Recall that SC2 of $u(x,s)$ (condition C3.1-A) is the necessary and sufficient condition for the choice of x which maximizes $u(x,s)$ (under certainty) to be nondecreasing in s. Thus, Corollary 3.1 gives necessary and sufficient conditions for the preservation of comparative statics results under uncertainty.[28] Any result that holds when s is known, will hold when s is unknown but the distribution of s experiences an MLR shift. Further, MLR shifts are the weakest distributional shifts that guarantee that conclusion. The result is illustrated with two examples.

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[26] Building from a paper by Eeckhoudt and Gollier (1995) for portfolio problems, Athey (1998) uses the techniques of this paper to extend this result to incorporate the restriction that investors are also risk averse; under that condition, it follows that the restriction that f is log-spm can be weakened to allow that only the distribution, F, is log-spm.

[27] The quantification over constraint sets B can be dropped for the conclusion that (B) is necessary.

[28] Jewitt (1987) and Ormiston and Schlee (1993) also give this interpretation in their analyses. Ormiston and Schlee (1993) explicitly analyze the preservation of comparative statics with respect to MLR shifts, and further show, under additional regularity assumptions, that single crossing of u is a necessary condition for the result to hold for all MLR shifts.

4.2.1. The Choice of Distribution and Changes in Risk Preferences

This section considers the consequences for Corollary 3.1 of placing additional assumptions on risk preferences. The context is a problem where the choice variable is a parameter of a probability distribution (for example, the agent chooses effort in a principal-agent problem, or makes an investment decision), and the exogenous parameter describes risk preferences. Corollary 3.1 can then be applied systematically to provide succinct proofs of some existing results, and further to suggest some new ones.

Formally, let $u(s,q)$ be an agent's utility function, and let $f(s;x)$ be a density with an associated probability distribution $F(s;x)=\int_{-\infty}^{s} f(t;x)dm(t)$. The agent solves $\max_{x\in B}\int_{s} u(s,q)f(s;x)dm(s)$. This is an example where it is useful to have results that do not rely on concavity of the objective: concavity of the objective in x requires additional assumptions (see Jewitt (1988) and Athey (1995)), which may or may not be reasonable in a given application.

We will now study several sets of sufficient conditions for the agent's objective to satisfy SC2 in $(x;q)$, each corresponding to different classes of applications. For simplicity, consider the case where u satisfies enough regularity conditions so that the integration by parts is valid. Then:[29]

$$\arg\max_{x\in B}\int_{s} u(s,q)f(s;x)dm(s) = \arg\max_{x\in B}\; -\int_{s} u_s(s,q)F(s;x)ds$$
$$=\arg\max_{x\in B}\; u_s(\bar{s},q)\int_{s} s f(s;x)ds + \int_{s} u_{ss}(s,q)\int_{t=\underline{s}}^{s} F(t;x)dt\, ds$$

The following result follows directly from these equations and Corollary 3.1. Of course, it can be further extended to allow for restrictions on higher order derivatives.

Proposition 3 *Consider the following conclusion: (C) $x^*(q,B)=\arg\max_{x\in B}\int_{s} u(s,q)f(s;x)dm(s)$ is nondecreasing q for all B. In each of the following, (A) and (B) are a MPSC for (C):*

	Additional Assumptions	*A*	*B*
(i)	$u(s,q)\geq 0$, *and* $\{s \mid u(s,q)\neq 0\}$ *is constant in q.*	$u(s,q)$ *is log-spm. a.e.-m.*	$f(s;x)$ *is WSC2 in* $(x;s)$ *a.e.-m.*
(ii)	$u_s(s,q)\geq 0$, *and* $\{s \mid u_s(s,q)\neq 0\}$ *is constant in q.*	$u_s(s,q)$ *is log-spm. in* $(s,-q)$ *a.e.-m.*	$F(s;x)$ *is WSC2 in* $(x;s)$ *a.e.-Lb.*
(iii)	$\int_{s} s f(s;x)ds$ *is constant in x,* $u_{ss}\leq 0$, *and* $\{s \mid u_{ss}(s,q)\neq 0\}$ *is constant in q.*	$\lvert u_{ss}(s,q)\rvert$ *is log-spm. in* $(s,-q)$ *a.e.-m.*	$\int_{t=-\infty}^{s} F(t;x)dt$ *is WSC2 in* $(x;s)$ *a.e.-Lb.*

In each of (i)-(iii), the fact that (B) is necessary for the conclusion to hold whenever (A) holds relies crucially on the nonmonotonicity of the relevant function in (A). Thus, while the sufficient conditions and the necessity of (A) in each case are quite general, one should be more careful in drawing conclusions about the necessity of (B). However, there are applications that are appropriate for each of these results. Consider each of (i)-(iii) in turn.

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29 The notation $\underline{s}$ and $\bar{s}$ indicates the bounds of the support of F.

To begin, compare (i) with Corollary 3.1: the single crossing condition on the payoff to the project is replaced by a single crossing condition on the density. Case (i) might apply if a principal is restricted to offer a stochastic mechanism to an agent, and the uncertainty is about the allocation that will be received.[30] The payoff u is log-spm when higher types have a larger relative return to s. When the support of s is fixed, WSC2 of a probability density is equivalent to weak single crossing of ratios (WSCR), which is stronger than FOSD but weaker than the MLR.

Now, consider Part (ii). Hypothesis (A) requires that the agent's Arrow-Pratt risk aversion is nondecreasing in q (i.e., u_s is log-spm in $(s,-q)$). Further, (B) requires that the distribution $F(s;x)$ satisfies WSC2 in $(x;s)$. That is, $F(s;x_H)$ crosses $F(s;x_L)$ at most once, from below. Under this assumption, it is possible that increasing x decreases the mean as well as the riskiness of the distribution; that is, it might incorporate a mean-risk tradeoff.

In case (iii), the agents are restricted to be risk averse. We see that x always increases with an agent's prudence (as defined in Section 3.1) if and only if $\int_{-\infty}^{s} F(v,x)dv$ satisfies WSC2 in $(x;s)$.

Of these results, only (ii) has received attention in the literature. Diamond and Stiglitz (1974) established the sufficiency side of the relationship, and many authors have since exploited and further studied the result (such as Jewitt (1987, 1989)). The example further illustrates how Jewitt's (1987) analysis of risk aversion relates to this paper: Jewitt (1987) shows that (A) is necessary and sufficient for (C) to hold whenever (B) does.

4.2.2. Mineral Rights Auction with Asymmetries and Risk Aversion

This section studies existence of a PSNE in nondecreasing strategies in Milgrom and Weber's (1982) model of a mineral rights auction, generalized to allow for risk averse, asymmetric bidders whose utility functions are not necessarily differentiable, reserve prices may differ by bidders, and when bidding units may be discrete.[31]

The analysis begins with the case where there are two bidders; difficulties will arise when we consider n asymmetric bidders. Suppose that bidders one and two observe signals s_1 and s_2, respectively, where each agent's utility (written $u_i(b_i,s_1,s_2)$) satisfies

$$u_i(b_i,s_1,s_2) \text{ is nondecreasing in } (-b_i,s_1,s_2) \text{ and supermodular in } (b_i,s_j),\ j=1,2. \qquad (4.4)$$

The signals have a joint density $h(s_1,s_2)$ with respect to Lebesgue measure, and the conditional density of s_{-i} given s_i is written $f_{-i}(s_{-i}|s_i)$. To see an example where assumption (4.4) is satisfied, let $u_i(b_i,s_1,s_2)=\int \hat{u}_i(v-b_i)g(v|s_1,s_2)dv$, where v is affiliated with s_1 and s_2, and $\hat{u}_i$ is nondecreasing and concave.[32]

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30 Such uncertainty might arise if the principal cannot observe the agent's choice perfectly, or if the principal must design an error-prone bureaucratic system to carry out the regulation.

31 As in the pricing game studied in Section 2, this is *not* a game with strategic complementarities between players bidding functions, so Vives (1990) may not be applied to establish existence of a PSNE.

32 To see why, note that s_i and s_j each induce a first order stochastic dominance shift on F, and $\hat{u}_i$ is supermodular

When player two uses the bidding function $b(s_2)$, then the set of best reply bids for player one given her signal (s_1) can be written (assuming ties are broken randomly):

$$b_1^*(s_1) = \arg\max_{b_1 \in B} \int_{s_2} u_1(b_1, s_1, s_2)\mathbf{1}_{b_1 > b(s_2)}(s_2)\, f_2(s_2|s_1)ds_2 + \tfrac{1}{2}\int_{s_2} u_1(b_1, s_1, s_2)\mathbf{1}_{b_1 = b(s_2)}(s_2)\, f_2(s_2|s_1)ds_2$$

When bidder two plays a nondecreasing strategy, (4.4) implies that bidder one's payoff function given a realization of s_2 satisfies WSC2 in $(b_1; s_2)$. The returns to increasing the bid from b_L to b_H are strictly negative for low values of s_2, when the opponent bids less than b_L; the returns are increasing in s_2 on the region where raising the bid causes the player to win, where she would have lost with b_L; and the effect is zero for s_2 so high that even b_H does not win. For each bid, the returns are increasing in s_2. This yields (using Extension (ii) to Theorem 3, and Corollary 3.1):

Proposition 3 *Consider the 2-bidder mineral rights model, where the utility function satisfies 4.4 above, the joint density h is log-spm almost everywhere (affiliated), and the support of the random variables is fixed. Suppose that bidder 2 uses a strategy* $b_2(s_2)$ *which is nondecreasing in* s_2*. Then* $b_1^*(s_1)$ *is nondecreasing (strong set order) in* s_1.

Athey (1997) shows that in auction games such as the first price auction described above, a PSNE exists if the conclusion of Proposition 3 holds, and further, the type distributions are atomless and affiliated, and either the players choose from a finite set of bids (i.e., bidding in pennies) or else u_i is continuous.

What happens when we try to extend this model to *n* bidders? If the bidders face a symmetric distribution, and all opponents use the same symmetric bidding function, then only the maximum signal of all of the opponents will be relevant to bidder one. Define $s_m = \max(s_2,..,s_n)$. Milgrom and Weber (1982) show that (s_1,s_m) are affiliated when the distribution is symmetric. Further, if the opponents are using the same strategies, whichever opponent has the highest signal will necessarily have the highest bid. Then we can apply Proposition 3 to this problem exactly as if there were only two bidders. Unfortunately, the results do not in general extend to *n*-bidder, asymmetric auctions with common value elements. Under asymmetric distributions (or if players use asymmetric strategies), affiliation of the signals is not sufficient to guarantee that the signal of the highest bidder is affiliated with a given player's signal.

The conclusions we can draw about the equilibria of auction games based on Proposition 3 are stronger than those available in the existing literature, which has typically considered only continuous bidding units, and either symmetry or stronger distributional assumptions, such as private values or independent signals.[33]

4.3. SC3 and The Spence-Mirrlees Single Crossing Property (SC3)

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in (b_i,v). By Athey (1995), supermodularity of the expectation in (b_i,s_i) and (b_i,s_j) follows.

[33] See Lebrun (1996), who uses a differential equations approach to derive existence in an asymmetric independent private values auction.

This section extends the results about single crossing to consider single crossing of indifference curves, providing general comparative statics theorems and applications to an education signaling game and a consumption-savings problem. The SM single crossing property is central to the analysis of monotonicity in standard signaling and screening games, as well as many other mechanism design problems. The SM condition for an arbitrary differentiable function $h:\Re^3 \to \Re$ that satisfies $\frac{\partial}{\partial y}h(x,y,t) \neq 0$ is given as follows:

$\frac{\partial}{\partial x}h(x,y,t) \Big/ \left|\frac{\partial}{\partial y}h(x,y,t)\right|$ is nondecreasing in t. (SM)

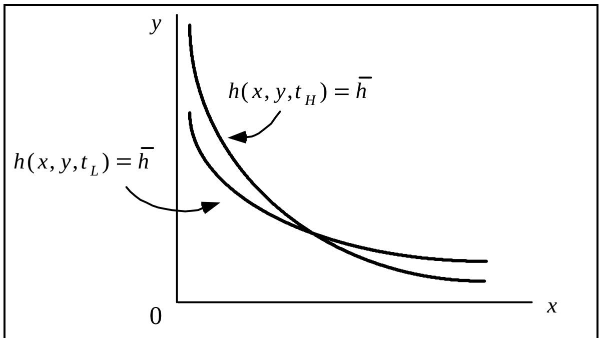

Figure 2: $h(x,y,t)$ satisfies SC3.

When the (x,y) indifference curves are well defined, SM is equivalent to the requirement that the indifference curves cross at most once as a function of t (Figure 2). We will make use the following assumption which guarantees that the (x,y)-indifference curves are well-behaved (it is also possible to generalize SM to the case where h is not differentiable, but we maintain (WB) for simplicity):

h is differentiable in (x,y); $\frac{\partial}{\partial y}h(x,y,t) \neq 0$; the (x,y)-indifference curves are closed curves. (WB)

Milgrom and Shannon (1994) show that under (WB), (SM) is equivalent to SC3 (Definition 1).

This section characterizes the SM single crossing property for objective functions of the form $V(x,y,q) \equiv \int_s v(x,y,s)\, f(s;q)dm(s)$, with the following theorem (analogous to Theorem 3).

Theorem 6 *Assume that $v:\Re^3 \to \Re$ satisfies (WB). Then (A) v(x,y,s) satisfies SC3 a.e.-m, and.(B) f(s;q) is log-spm a.e.-m, are a MPSC for (C) $\int_s v(x,y,s)\, f(s;q)dm(s)$ satisfies SC3.*

Corollary 6.1 (Comparative Statics and the Spence-Mirlees SCP) *Theorem 6 also holds if 4.7-C is replaced with (C) $x^*(\theta,B)$= arg $\max_{x\in B}\int_s v(x,b(x),s)\, f(s;q)dm(s)$ is nondecreasing in q for all B and all b:$\Re\to\Re$.*

Theorem 6 is closely related to Theorem 3, although the class of payoff functions considered is more restrictive. Sufficiency in Theorem 6 can also be shown using Lemma 2. Let: $h_1(s)=|v_y(x,y,s)|\, f(s;q_L)$, $h_2(s)=v_x(x,y,s)\, f(s;q_H)$ $h_3(s)=v_x(x,y,s)\, f(s;q_H)$, $h_4(s)=|v_y(x,y,s)|\, f(s;q_L)$, and note that A and B imply:

$$\frac{v_x(x,y,s_L)}{|v_y(x,y,s_L)|}\frac{f(s_H;q_L)}{f(s_L;q_L)} \leq \frac{v_x(x,y,s_H)}{|v_y(x,y,s_H)|}\frac{f(s_H;q_H)}{f(s_L;q_H)} \qquad (4.5)$$

This in turn implies by Lemma 2 that $\frac{\partial}{\partial x}V(x,y,q_L)\Big/\left|\frac{\partial}{\partial y}V(x,y,q_L)\right| \leq \frac{\partial}{\partial x}V(x,y,q_H)\Big/\left|\frac{\partial}{\partial y}V(x,y,q_H)\right|$.

This theorem can be applied to an education signaling model, where x represents a worker's choice of education, y is monetary income, and q is a noisy signal of the worker's ability, s (for example, the workers' experience in previous schooling). If the worker's preferences $u(x,y,s)$

satisfy SM and higher signals increase the likelihood of high ability, the worker's education choice will be nondecreasing in the signal q for any wage function $w(x)$.

In another example, consider a variation on the standard consumption-savings problem. Let z denote an agent's initial wealth, and let x denote savings and $b(x)$ the (endogenously determined) value function of savings. The agent further has a non-tradable endowment (q) of a risky asset (s), where the value of s is unknown at the time that the savings decision is made. The probability distribution over s at the time the savings decision is made is given by $F(s;q)$. The parameter q might represent the quality or quantity of an exogenously (or previously) determined endowment. The agent's utility given a realization of s is $u(z+s-x,b(x))$, which is assumed to be nondecreasing. Then, our agent solves $\max_{x\in[0,z]}\int u(z+s-x,b(x))dF(s;q)$. Then, the following two conditions are a minimal pair of sufficient conditions for the comparative statics conclusion that savings increases in q: (A) the marginal rate of substitution of current for future utility, u_1/u_2, is nondecreasing in s, and (B) F satisfies MLR. Thus, a FOSD improvement in the distribution over the consumption good does not necessarily increase savings.

5. Conclusions

The goal of this paper has been to study systematic conditions for comparative statics predictions in stochastic problems, and to show how the results can be used in economic applications. As such, this paper derives a few main theorems (summarized in Table 1) which can be used to characterize properties of stochastic objective functions that arise frequently in the context of comparative statics problems in economics. The properties considered in this paper, the various single crossing properties and log-supermodularity, are each necessary and sufficient for comparative statics in an appropriately defined class of problems. Several variations of these results are also analyzed, each of which is motivated by the requirements of different economic problems. Because the properties studied in this paper, and the corresponding comparative statics predictions, do not rely on differentiability or concavity, the results from this paper can be applied in a wider variety of economic contexts than similar results from the existing literature.

It turns out that a few results from the statistics literature are at work behind the sufficient conditions for all of the properties studied in this paper. This is especially surprising because the statistics literature did not emphasize comparative statics. However, the common structure underlying the results in this paper provide a useful perspective on the relationships between seemingly unrelated problems. Building on the results from statistics, this paper characterizes the conditions which economists can use directly for comparative statics predictions.

Finally, this paper focuses on results about necessity as well as their limitations, sharply defining the tradeoffs that must occur between weakening and strengthening assumptions about various components of economic models. These tradeoffs are highlighted in Table 1, which further provides guidance about which properties will be most appropriate in different classes of problems. Thus, together with Athey (1995)'s analysis of stochastic supermodularity, concavity, and other differential properties, the results in this paper can be used to identify exactly which assumptions are the right ones to guarantee robust monotone comparative statics predictions in a wide variety of stochastic problems.

Table 1: Summary of Results

Thm #	A: Hypothesis on u (a.e.-m)	B: Hypothesis on f (a.e.-m)	C: Conclusion	MCS: Equivalent Comparative Statics Conclusion
Thm 1; Cor 1.1	$u(\mathbf{x},\mathbf{s})\geq 0$ is log-spm.	$f(\mathbf{s},\boldsymbol{\theta})$ is log-spm.	$\int u(\mathbf{x},\mathbf{s})f(\mathbf{s},\boldsymbol{\theta})dm(\mathbf{s})$ is log-spm. in $(\mathbf{x},\boldsymbol{\theta})$.	$\arg\max_{x\in B}\int u(\mathbf{x},\mathbf{s})f(\mathbf{s},\boldsymbol{\theta})dm(\mathbf{s})$ $\uparrow$ in $(\boldsymbol{\theta},B)$.
Thm 2; Cor 2.1	$u(\mathbf{x},\mathbf{s})$ is spm. in $(\mathbf{x},s_i)$ $\forall$ i.	$f(\mathbf{s},\boldsymbol{\theta})$ is log-spm.	$\int_A u(\mathbf{x},\mathbf{s})\frac{f(\mathbf{s},q)}{\int_A f(\mathbf{s},q)d\mathbf{s}}dm(\mathbf{s})$ is spm. in $(\mathbf{x},\boldsymbol{\theta},A)$.	$\arg\max_{x\in B}\int_A u(\mathbf{x},\mathbf{s})\frac{f(\mathbf{s},q)}{\int_A f(\mathbf{s},q)d\mathbf{s}}dm(\mathbf{s})$ $\uparrow$ in $(\boldsymbol{\theta},A,B)$.
Thm 3; Cor 3.1	$u(x,s)$ satisfies SC2 in $(x;s)$.	$f(s,q)$ is log-spm.	$\int u(x,s)f(s,q)dm(s)$ satisfies SC2 in $(x;q)$.	$\arg\max_{x\in B}\int u(x,s)f(s,q)\, dm(s)$ $\uparrow$ in q for all B.
Thm 6, Cor 6.1	$u(x,y,s)$ satisfies SC3.	$f(s,q)$ is log-spm.	$\int u(x,y,s)f(s,q)dm(s)$ satisfies SC3.	$\arg\max_{x\in B}\int u(x,b(x),s)f(s,q)dm(s)$ $\uparrow$ in q for all $b:\Re\to\Re$.
	Extensions: assume, in addition, that $\text{supp}[F(s;q)]$ is constant in q.			
Thm 4	$u(x,s)$ satisfies WSC2(s_0).	$f(s,q)\geq 0$ satisfies WSCR(s_0).	$\int u(x,s)f(s,q)dm(s)$ satisfies SC2 in $(x;q)$.	$\arg\max_{x\in B}\int u(x,s)f(s,q)\, dm(s)$ $\uparrow$ in q for all B.
Thm 5	$u(x,s)$ satisfies WSCR(s_0).	$f(s,q)$ is log-spm.	$\int u(x,s)f(s,q)dm(s)$ satisfies SC2 in $(x;q)$.	$\arg\max_{x\in B}\int u(x,s)f(s,q)\, dm(s)$ $\uparrow$ in q for all B.

In each row: A and B are a *minimal pair of sufficient conditions* (Definition 5) for the conclusion C; further, C is equivalent to the comparative statics result in column 4.

Notation and Definitions: Bold variables are vectors in $\Re^n$; italicized variables are real numbers; f is non-negative; spm. indicates supermodular, and log-spm. indicates log-supermodular (Definition 2); sets are increasing in the strong set order (Definition 4); SC2 indicates single crossing of incremental returns to x, and SC3 indicates single crossing of x-y indifference curves (Definition 1); WSC2(s_0) indicates weak SC2 with a fixed crossing point, s_0 (Definition 1); WSCR(s_0) indicates weak single crossing of ratios at a fixed crossing point s_0 (Definition 5). Arrows indicate weak monotonicity.

6. Appendix

Proof of Lemma 4 and Theorem 1: Sufficiency in Theorem 1 follows from Lemma 2; sufficiency in Lemma 4 will follow from Lemma 2 since the test functions are log-spm by Lemma 3. Necessity in Lemma 4 will be established below in (1) and (2), noticing that our counter-examples come from the relevant set. Necessity in Lemma 4 implies necessity in Theorem 1. We will treat necessity for *f*, the argument for *u* is of course analogous:

(1) n=1: Let $n(A|q)$=$\int_A f(s;q)dm(s)$. Pick any two intervals of length *e*, such that $S_H(e) \geq S_L(e)$ and $S_H(e) \cap S_L(e) = \emptyset$. Let $u(x_L,s)$ = $\mathbf{1}_{S_L(e)}(s)$, and let $u(x_H,s)$ = $\mathbf{1}_{S_H(e)}(s)$. Then $\int u(x_L,s)f(s,q)dm(s) = n(S_L|q)$ and $\int u(x_H,s)f(s,q)dm(s) = n(S_H|q)$. Since *u* is log-spm, then $\int u(x,s)f(s,q)dm(s)$ must be log-spm by (C), i.e., $n(S_H,q_H)n(S_L,q_L) \geq n(S_L,q_H)n(S_H,q_L)$. Standard limiting arguments (i.e. Martingale convergence theorem as applied in the appendix of Milgrom and Weber (1982)) imply that $f(s,q)$ must be log-spm *m*-almost everywhere.

(2) $n \geq 2$: Let $n(A|q)$=$\int_A f(\mathbf{s};\ )dm(\mathbf{s})$. We partition $\Re^n$ into *n*-cubes of the form $[\ _1 - \frac{1}{2^t}, (i_1 +\ )\frac{1}{2^t}] \times \cdot\cdot [i_n - \frac{1}{2^t}, (\ _n + 1)\frac{1}{2^t}]$, and let $Q^t(\mathbf{s})$ be the unique cube containing **s**. Consider a *t*, and take any $\mathbf{a},\mathbf{b} \in \Re^n$. Further, let $X = \{\mathbf{y}, \mathbf{z}, \mathbf{y} \vee \mathbf{z}, \mathbf{y} \wedge \mathbf{z}\}$, where each element of *X* is distinct. Define $u(\mathbf{x},\mathbf{s})$ on *X* as follows: $u(\mathbf{y},\mathbf{s})$ = $\ _{Q^t(\mathbf{a})}(\mathbf{s}\ $, $u(\mathbf{z},\mathbf{s})$ = $\mathbf{1}_{Q^t(\mathbf{b})}(\mathbf{s})$, $u\ \mathbf{y} \vee \mathbf{z},\ )$ = $\mathbf{1}_{Q^t(\mathbf{a} \vee \mathbf{b})}(\mathbf{s})$, and $u\ \mathbf{y} \wedge \mathbf{z},\ )$ = $\mathbf{1}_{Q^t(\mathbf{a} \wedge \mathbf{b})}(\mathbf{s})$ *u* is log-spm. If $\int u(\mathbf{x},\mathbf{s})f(\mathbf{s},\ )dm(\mathbf{s})$ is log-spm, it follows that $n(Q^t(\mathbf{a} \vee \mathbf{b})|\ \ \vee\ \ ')\cdot n(Q^t(\mathbf{a} \wedge \mathbf{b})|\ \ \vee\ \ ')\ \geq n(Q^t(\mathbf{a})|\ \ )\cdot n(Q^t(\mathbf{b})|\ \ ')$ for all θ,θ'. Since this must hold for all *t* and for all **a**,**b**, we can use the Martingale convergence theorem to conclude that $f(\mathbf{a} \vee \mathbf{b})|\ \ \vee\ \ ')\cdot f(\mathbf{a} \wedge \mathbf{b})|\ \ \wedge\ \ ') \geq f(\mathbf{a}|\ \ )\cdot f(\mathbf{b}|\ \ ')$ for *m*-almost all **a**,**b** (recalling that *m* is a product measure).

Proof of Theorem 2: (A) and (B) imply (C): Sufficiency is shown in the text for $g \geq 0$. If *g* is bounded below, let $M = \left|\inf_{\mathbf{s} \in S}\{g(\mathbf{s})\}\right|$; otherwise, let *M* be an arbitrary positive constant, and apply the same arguments with $g(\mathbf{s})+M$, taking the limit as *M* approaches ∞ if *g* is unbounded below.

(A) and (C) imply (B): Consider **a**,**b** such that $n^L(Q^t(\mathbf{b}))\cdot n^H(Q^t(\mathbf{a})) > 0$ (otherwise there is nothing to check) and such that each element of $\{\mathbf{a},\mathbf{b},\mathbf{a} \vee \mathbf{b},\mathbf{a} \wedge \mathbf{b}\}$ is distinct. Let $Q^t(\mathbf{s})$ be the unique cube containing **s** defined in the proof of Theorem 1, and let , and let $A_H = \{Q^t(\mathbf{a}),Q^t(\mathbf{a} \vee \mathbf{b})\}$ and $A_L = \{Q^t(\mathbf{b}),Q^t(\mathbf{a} \wedge \mathbf{b})\}$. Choose $g(\mathbf{s}) = \mathbf{1}_C(\mathbf{s})$ so that $Q^t(\mathbf{a} \vee \mathbf{b}),Q^t(\mathbf{b}) \subseteq C$ and $Q^t(\mathbf{a} \wedge \mathbf{b}),Q^t(\mathbf{a}) \not\subset C$, and further so that $\mathbf{1}_C(\mathbf{s})$ is nondecreasing. Then, $G(\cdot|A_H,q_H) > G(\cdot|A_L,q_H)$ if and only if $\int_{\mathbf{s} \in A^L} g(\mathbf{s})f(\mathbf{s};q_L)dm(\mathbf{s})\cdot\int_{\mathbf{s} \in A^H} f(\mathbf{s};q_H)dm(\mathbf{s}) \leq \int_{\mathbf{s} \in A^H} g(\mathbf{s})f(\mathbf{s};q_H)dm(\mathbf{s})\cdot\int_{\mathbf{s} \in A^L} f(\mathbf{s};q_L)dm(\mathbf{s})$, which reduces to:

$$n^L(Q^t(\mathbf{b}))\cdot\left[n^H(Q^t(\mathbf{a} \vee \mathbf{b})) + n^H(Q^t(\mathbf{a}))\right] \leq n^H(Q^t(\mathbf{a} \vee \mathbf{b}))\cdot\left[n^L(Q^t(\mathbf{b})) + n^L(Q^t(\mathbf{a} \wedge \mathbf{b}))\right].$$

Simplifying gives $n^L(Q^t(\mathbf{b}))\cdot n^H(Q^t(\mathbf{a})) \leq n^H(Q^t(\mathbf{a} \vee \mathbf{b}))\cdot n^L(Q^t(\mathbf{a} \wedge \mathbf{b}))$. Since this must hold for all *t* and all **a**, **b**, we conclude that $f(\mathbf{s},q)$ must be log-spm for *m*-almost all **a**,**b** and all *q*.

(B) and (C) imply (A): Consider $\mathbf{a} < \mathbf{b}$ and let $A_L = Q^t(\mathbf{a})$ and $A_H = Q^t(\mathbf{b})$. Then $G(\cdot|A_H) > G(\cdot|A_L)$ if and only if $\int_{\mathbf{s} \in Q^t(\mathbf{a})} g(\mathbf{s})f(\mathbf{s}|q,Q^t(\mathbf{a}))dm(\mathbf{s}) \leq \int_{\mathbf{s} \in Q^t(\mathbf{b})} g(\mathbf{s})f(\mathbf{s}|q,Q^t(\mathbf{b}))dm(\mathbf{s})$, which is true for all $\mathbf{a} < \mathbf{b}$ and all *t* if and only if $g(\mathbf{a}) \leq g(\mathbf{b})$ for *m* -almost all $\mathbf{a} < \mathbf{b}$.

Proof of Corollary 2.2: (ii) implies (i): Since $[1/F(a;q)]\int_{-\infty}^{s} \mathbf{1}_{t \leq a}(t)dF(t;q)$ is a probability distribution for all a and q, we can apply a result from Athey (1995): $G(q|a)$ is nondecreasing in q and a and all g nondecreasing, if and only if $G(q|a)$ is nondecreasing in q for all a, all $g(s)=\mathbf{1}_{s\geq b}(s)$, and all b. Rewriting the latter condition, we have $[1/F(a;q)]\int_b^a dF(s;q)$ nondecreasing in q and a for all $b<a$, which is true if and only if $1-F(b;q)/F(a;q)$ is nondecreasing in q for all $b<a$. Log-spm of F and the fact that F is a distribution yield the result.

Likewise, $G(\cdot|a)$ is nondecreasing in a for all g nondecreasing if and only if $G(q|a)$ is nondecreasing in a for all $g(s)=\mathbf{1}_{s\geq b}(s)$ and all $b<a$. From above, we check that $1-F(b;q)/F(a;q)$ is nondecreasing in a, which is true since F is a distribution.

(i) implies (ii): Simply apply (i) for all $b<a$, which implies that $1-F(b;q)/F(a;q)$ is nondecreasing in q for all $b<a$. This is equivalent to log-spm of F.

Proof of Lemma 6: Pick $q_H > q_L$. Define measures n_L and n_H as follows: $n_L(A)=\int_A dK(s;q_L)$ and $n_H(A) = \int_A dK(s;q_H)$. Define $a = \inf\{s|s \in \text{supp}[K(\cdot;q_H)]\}$ and $b = \sup\{s|s \in \text{supp}[K(\cdot;q_L)]\}$. Define $h(s;q)\equiv dK(s;q_H)/dC(s;q_H,q_L)$. Let $D\equiv\text{supp}[K(\cdot;q_L)]\cup\text{supp}[K(\cdot;q_H)]$. Since the behavior of $h(s;q)$ outside of D will not matter, we will restrict attention to D. The proof proceeds in several steps.

Part (a): If $a\geq b$, then the conclusions hold automatically. Throughout the rest of the proof, we treat the case where $a < b$.

Part (b): For any $S \equiv (s_L, s_H] \subset [a,b]$, $n_L(S) > 0$ implies that $n_H(S) > 0$. Proof: Suppose that $n_L(S) > 0$ and $n_H(S) = 0$. Note that $0 < K(s_L;q_H)$ since $a<s_L$. If $\text{supp}[K(s_L;q_H)]\leq s_L$, then define g as follows. $g(s)=-1$ for $s \in (-\infty, s_L)$, while $g(s)=K(s_L;q_L)/n_L([s_L,\infty))$ for $s \in [s_L,\infty)$. Otherwise, define g as follows: $g(s)=-1$ for $s \in (-\infty, s_L)$, $g(s)=K(s_L;q_L)/n_L(S)$ for $s\in S$, and $g(s)=.9\cdot K(s_L;q_H)/n_H([s_H,\infty))$ for $s \in [s_H,\infty)$. Now, it is straightforward to verify that SC1 is violated for this g.

Part (c): For any $S \equiv (s_L, s_H] \subset [a,b]$, $n_H(S) > 0$ implies that $n_L(S) > 0$. Proof: Suppose that $n_H(S) > 0$ and $n_L(S) = 0$. If $K(s_L;q_L)=0$, then define g as follows: $g(s)=-1$ for $s \in (-\infty, s_H)$, while $g(s)=K(s_H;q_H)/(2n_H([s_H,\infty))$ for $s \in [s_H,\infty)$. Otherwise, define g as follows: $g(s)=-n_L([s_H,\infty))/K(s_L;q_L)$ for $s \in (-\infty, s_L)$, $g(s)=-n_H([s_H,\infty))/n_H(S)$ for $s\in S$, and $g(s)=1$ for $s \in [s_H,\infty)$. Now, it is straightforward to verify that SC1 is violated for this g.

Part (d): $\text{supp}[K(\cdot;q_H)]\geq\text{supp}[K(\cdot;q_L)]$ in the strong set order. Proof: Parts (c) and (d) imply that n_H is absolutely continuous with respect to n_L on $[a,b]$, and vice versa. It now suffices to show that if $s' \in \text{supp}[K(\cdot;q_H)]$ and $s''\in \text{supp}[K(\cdot;q_L)]$, and $s''>s'$, then $s',s''\in \text{supp}[K(\cdot;q_L)]\cap\text{supp}[K(\cdot;q_H)]$. Since $s\notin \text{supp}[K(\cdot;q_L)]$ for all $s>b$, we may restrict attention to $s''\leq b$. Likewise we may restrict attention to $s'>a$. But, if (as we argued in part (b)) n_H is absolutely continuous with respect to n_L and vice versa on $[a,b]$, then $\text{supp}[K(\cdot;q_H)]=\text{supp}[K(\cdot;q_L)]$ on $[a,b]$, and we are done.

Lemma A1 $G(q)=\int g(s;q)k(s;q)dm(s)$ *satisfies SC1 in q under the following sufficient conditions: (i)(a) For each q, g(s,q) satisfies WSC1 in s a.e.-m; for m-almost all s, g(s,q) is nondecreasing in q. (i)(b) k(s;q) is log-spm a.e-m. (i)(c) Either g(s,q) satisfies SC1 in s a.e-m, or else* $\text{supp}[K(\cdot;q)]$ *is constant in q.*

Proof of Lemma A1: Pick $q_H > q_L$. Suppose that $\int g(s;q_L)k(s;q_L)dm(s) \geq(>)0$. Choose s_0 so that $g(s,q_L) \geq 0$ for *m*- almost all $s > s_0$ and $g(s,q_L) \leq 0$ for *m*- almost all $s \leq s_0$ (s_0 exists by the definition of WSC1). Let $\hat{s}_0 = \min\{s \geq s_0 \text{ and } s \in \text{supp}[K(\cdot;q_L)]\}$. First, note that if $\hat{s}_0 \notin$ supp[$K(\cdot;q_H)$], then supp[$K(\cdot;q_H)$]> $\hat{s}_0$ since supp[$K(\cdot;q_H)$] is nondecreasing in the strong set order (applying Lemma 6 and (i)(b)). This in turn implies that, for all *s* in supp[$K(\cdot;q_H)$], $g(s,q_H) \geq g(s,q_L) \geq 0$, which implies $G(q_H)\geq 0$. Further, if $G(q_L)>0$, then we can conclude that $G(q_H)>0$ by (i)(c). Second, observe that the case where supp[$K(\cdot;q_H)$]$\leq \hat{s}_0$ is degenerate, since this would imply by the strong set order that supp[$K(\cdot;q_L)$]$\leq \hat{s}_0$ as well. But then our hypothesis that $G(q_L)$ is nonnegative would imply that $g(s,q_L)=0$ a.e.-*m* on supp[$K(\cdot;q_L)$]. Since supp[$K(\cdot;q)$] is nondecreasing in the strong set order, this in turn implies $G(q_H)=0$ by (i)(c).

So, we consider the third case where $\hat{s}_0 \in$ supp[$K(\cdot;q_H)$], but there exist $s', s'' \in$ supp[$K(\cdot;q_H)$] such that $s' < \hat{s}_0 < s''$. Notice that, by the strong set order, supp[$K(\cdot;q_H)$]=supp[$K(\cdot;q_L)$] on an interval surrounding $\hat{s}_0$. It further implies that $g(s,q_L)\leq 0$ for all $s \in$ supp[$K(\cdot;q_L)$]\supp[$K(\cdot;q_H)$] (because everything in the set lies below supp[$K(\cdot;q_H)$], which contains $\hat{s}_0$). By the same reasoning, $g(s,q_L)\geq 0$ on supp[$K(\cdot;q_H)$]\supp[$K(\cdot;q_L)$].

Now, define a modified likelihood ratio $\hat{l}(s)$, as follows: $\hat{l}(s)=0$ for $s\notin$ supp[$K(\cdot;q_L)$], $\hat{l}(s)= k(s;q_H)/k(s;q_L)$ for $s\in$ supp[$K(\cdot;q_L)$] and $k(s;q_L)>0$, and then extend the function so that $\hat{l}(s) = \max\left(\lim_{s'\uparrow s} \hat{l}(s'), \lim_{s'\downarrow s} \hat{l}(s')\right)$ for $s\in$ supp[$K(\cdot;q_L)$] and $k(s;q_L)=0$ (recalling that the likelihood ratio can be assumed to be nondecreasing in *s* on supp[$K(\cdot;q_L)$] without loss of generality by (i)(b)). Thus, we know $\hat{l}(\hat{s}_0)>0$, since $\hat{s}_0 \in$ supp[$K(\cdot;q_L)$] and since supp[$K(\cdot;q_H)$]=supp[$K(\cdot;q_L)$] on an open interval surrounding $\hat{s}_0$. We use this to establish:

$$\begin{aligned}&\int g(s;q_H)k(s;q_H)dm(s) \geq \int g(s;q_L)k(s;q_H)dm(s) \geq \int g(s;q_L)\hat{l}(s)k(s;q_L)dm(s)\\ &\geq -l(\hat{s}_0)\int_{-\infty}^{\hat{s}_0} |g(s;q_L)|k(s;q_L)dm(s) + l(\hat{s}_0)\int_{\hat{s}_0}^{\infty} g(s;q_L)k(s;q_L)dm(s) = l(\hat{s}_0)\int g(s;q_L)k(s;q_L)dm(s)\end{aligned} \tag{6.1}$$

The first inequality follows by the fact that $g(s,q_H) \geq g(s,q_L)$. The second inequality follows by the definition of $\hat{l}(s)$ and since $g(s,q_L)\geq 0$ on supp[$K(\cdot;q_H)$]\supp[$K(\cdot;q_L)$]. The equality follows by WSC1 of *g*. The third inequality is true because $\hat{l}(s)$ is nondecreasing a.e.-*m* on supp[$K(\cdot;q_L)$] since *k* is log-spm. The last equality is definitional. Thus, since $\hat{l}(s)>0$, $G(q_L)\geq(>)0$ implies $G(q_H)\geq(>)0$.

Proof of Theorem 3 and Lemma 5: Sufficiency follows from Lemma A1. Necessity will follow by constructing counter-examples from the relevant sets. Necessity of (T3-A) follows by observing that for any $s_L<s_H$, if we let $S_L(e)= (s_L - e, s_L]$ and $S_H(e) = (s_H - e, s_H]$, the function $k(s,q^L)=\mathbf{1}_{S_L(e)}$, $k(s,q^H)=\mathbf{1}_{S_H(e)}$ is log-spm. Thus, (T3-C) will imply that *g* is SC1 a.e.-μ. Necessity of (T3-B): Define *a*, *b*, *h*, and *D* as in the proof of Lemma 6. It suffices to consider log-spm of *h*. First, notice that if $a \geq b$, then $h(s;q)$ is log-spm a.e.-*m*. To see this, observe that $a \geq b$ implies that $h(s;q_H) = 0$ and $h(s;q_L) = 2$ on supp[$K(\cdot;q_L)$], while $h(s;q_L) = 0$ and $h(s;q_H) = 2$ on supp[$K(\cdot;q_H)$]. This implies that $h(s;q)$ is log-spm.

Now, suppose $a<b$. Let $B =$ supp[$K(\cdot;q_L)$] $\cap$ supp[$K(\cdot;q_H)$], which we have shown is equivalent to $D\cap[a,b]$. Pick any $s_L,s_H\in B$, and define $S_L(e)= (s_L - e, s_L]$ and $S_H(e) = (s_H - e, s_H]$, such that

$s_H - e \geq s_L$. For the moment, we will supress the e in our notation. Suppose further that $s_L, s_H \in B$, but $n_H(S_H) \cdot n_L(S_L) < n_H(S_L) \cdot n_L(S_H)$. By definition, if $s_L > a$, then $n_H([a, s_L - e]) > 0$; then, by absolute continuity, $n_L([a, s_L - e]) > 0$ and thus $K(s_L - e; q_L) > 0$. Let $g(s;\delta)$ be defined as follows:

$$g(s;d) = \begin{cases} -d \cdot \dfrac{n_L([s_L,\infty)) - n_L(S_H)}{K(s_L - e; q_L)} & s \in (-\infty, s_L - e) \\ -1 & s \in S_L \end{cases} \qquad \begin{cases} d & s \in [s_L, \infty) \setminus S_H \\ \dfrac{n_L(S_L)}{n_L(S_H)} & s \in S_H \end{cases}$$

It is straightforward to verify that there exists a $\delta>0$ such that the single crossing property fails with this g. But this implies that for any e positive and in the relevant range, $n_H(S_H(e)) \cdot n_L(S_L(e)) \geq n_H(S_L(e)) \cdot n_L(S_H(e))$. But this implies that $h(s;q)$ is log-spm a.e.-C on B. Since $\text{supp}[K(\cdot;q_H)] \geq \text{supp}[K(\cdot;q_L)]$ in the strong set order, this implies that $h(s;q)$ is log-spm a.e.-C on D.

Proof of Theorem 6: (i): Under the assumptions of the theorem, $v(x,y,s)$ has SC3 if and only if $u(x,s;b) \equiv v(x,b(x),s)$ has SC2 in $(x;s)$ for all functions b. Furthermore, $V(x,y,q)$ has SC3 if and only if $U(x,q;b) \equiv V(x,b(x),q)$ has SC2 in $(x;q)$ for all functions b. So, if we know that $v(x,y,s)$ has SC3, then $u(x,s;b)$ has SC2 in $(x;s)$ for all functions b. If k is log-spm a.e.-m, then Theorem 3.1 implies that $U(x,q;b)$ has SC2 in $(x;s)$ for all functions b. But this in turn implies that $V(x,y,q)$ has SC3.

(ii): Consider any $f(s;q)$. Let $F(s;q) = \int_{-\infty}^{s} f(t;q)dm(t)$. The working paper shows that if $F(s;q)$ does not satisfy (MLR), then there exists a continuous g which satisfies SC1 so that $\int g(s)dF(s;q)$ fails SC1 (this is a continuous approximation to the test functions from Lemma 5). Consider this function g. We know that, since g is continuous and crosses zero only once, it must be monotone nondecreasing in some neighborhood of the crossing point/region. Now, define the following points, which are the boundaries of the region where $g(s) = 0$: $c \equiv \inf_{s\in\Re}\{s | g(s) = 0\}$, $d \equiv \sup_{s\in\Re}\{s | g(s) = 0\}$. Now, we can find a $d > 0$ and two corresponding points, $c_d \equiv \sup_{s<c}\{s | g(s) = -d\}$ and $d_d \equiv \inf_{s>d}\{s | g(s) = d\}$, such that $(c,d) \subset (c_d,d_d)$ and g is nondecreasing on (c_d,d_d). Let us define a new function, $a(s)$, as follows: $a(s) = d$ for $s \in (c_d,d_d)$, $a(s) = |g(s)|$ elsewhere. Now, pick any $x_H > x_L$, and let $v(x,y,s) = x \cdot g(s)/(x_H - x_L) + a(s) \cdot y$. Since $a(s)$ and $g(s)$ are continuous and $a(s) > 0$, v satisfies (WB). Finally, $\frac{\partial v}{\partial x} \Big/ \frac{\partial v}{\partial y} = g(s) / \left[(x_H - x_L) \cdot a(s)\right]$ is nondecreasing in s. Thus, v satisfies the assumptions of the theorem as well as SC3. Now, $\int v(x,y,s)dF(s;q)$ satisfies SC3 if and only if $\int v(x,b(x),s)dF(s;q)$ satisfies SC2 in $(x;q)$ for all b. Let $b(x) = 0$. But, $v(x_H,0,s) - v(x_L,0,s) = g(s)$, and by construction $\int g(s)dF(s;q)$ fails SC1, which in turn implies that $\int v(x,0,s)dF(s;q)$ fails SC2 in $(x;q)$. Thus, $\int v(x,y,s)dF(s;q)$ fails SC3.

(iii): If $v(x,y,s)$ fails the SC3, then there exists a $b(x)$ such that $v(x,b(x),s)$ fails SC2 in $(x;s)$. But then, Theorem 3 implies that there exists an $f(s;q)$ which is log-spm a.e.-m such that $\int v(x,b(x),s)f(s;q)dm(s)$ fails SC2 in $(x;q)$. We conclude that $\int v(x,y,s)f(s;q)dm(s)$ fails SC3.

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